AN ANALYTIC DETERMINATION OF STABILIZING FEEDBACK GAINS[1]

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Abstract

A new analytic method for the existence and determination of stabilizing gains for linear, time-invariant, single input, single output systems is derived. This method only requires a test of the sign pattern of a rational function at the real roots of a polynomial. An easily checkable necessary and sufficient condition for a polynomial to be a convex direction for a Hurwitz stable polynomial is obtained as a consequence of the main result.

1 Introduction

We consider the following old problem of control:

Given coprime polynomials $p(s)$, $q(s)$ with real coefficients, determine conditions under which a real number α exists such that $\phi(s,\alpha) = q(s) + \alpha p(s)$ has degree in s equal to the degree of q and is Hurwitz stable, i.e., has all its roots in the open left half complex plane. Determine the set of all such α if one exists.

Let us denote the set of real numbers by $\mathbf{R}$, the set of all Hurwitz stable polynomials by $\mathcal{H}$ and the degree in s of a nonzero polynomial p by $deg\, p$. If we define

$$A(p,q) := \{\alpha \in \mathbf{R} : \phi(s,\alpha) = q(s) + \alpha p(s) \in \mathcal{H}\ ,\ \ deg\,\phi = deg\, q\},$$

then the problem is to determine under what conditions $A(p,q) \neq \emptyset$ and to give a description of $A(p,q)$ if it is not empty.

There are several classical solutions to this problem. *Evans root-locus method* and *Nyquist stability criterion* are among the most widely used graphical solutions. The method of *Hurwitz determinants* and *Neimark D-decomposition* can be considered as non-graphical solutions.

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Let $n := deg\, q$ and $m := deg\, p$ and let q_n and p_m denote the coefficients of s^n and s^m in $q(s)$ and $p(s)$, respectively. Since

$$n \geq m, \tag{1}$$

is an obvious necessary condition for $A(p,q) \neq \emptyset$, we assume (1) in what follows. It is also clear that in the cases where $n = m$, $deg\, \phi(s, \alpha_0) < n$ for $\alpha_0 := -q_n/p_m$ so that α_0 is not an element of $A(p,q)$. We hence also assume below in this section that this point is excluded from the descriptions of $A(p,q)$ whenever $n = m$.

(i) *Evans root-locus method* [2]: The equation $\phi(s, \alpha) = 0$ implicitly defines a complex multiple-valued function $\alpha \mapsto s(\alpha)$. Evans root-locus is a plot of the values of this function in the complex plane parameterized by $\alpha \in (-\infty, \infty)$. Evans derived certain rules by which the root-locus can be mechanically constructed provided the roots of $q(s)$ and $p(s)$ are known. This has been mainly responsible for the popularity of the root-locus method in relation to the above problem until the present day of high-speed computation. Now the root-locus is plotted by a repetitive application of fast root finding algorithms [4]. Once the complete plot is determined, the set $A(p,q)$ is the set of values of α for which all values of the function $\alpha \mapsto s(\alpha)$ are in the open left half complex plane.

(ii) *Nyquist stability criterion* [9]: Let $\mathbf{R}[s]$ denote the set of real polynomials in s. Given $p, q \in \mathbf{R}[s]$ with $q(j\omega) \neq 0$ for any $\omega \in \mathbf{R}$, let

$$\frac{p(j\omega)}{q(j\omega)} = \tilde{H}(\omega) + j\tilde{G}(\omega), \tag{2}$$

where $\tilde{H}(\omega) := Re\,\{p(j\omega)/q(j\omega)\}$ and $\tilde{G}(\omega) := Im\,\{p(j\omega)/q(j\omega)\}$. The plot of $\tilde{H}(\omega)$ versus $\tilde{G}(\omega)$ in rectangular coordinates as ω increases from 0 to ∞ is called the Nyquist plot (or the frequency response plot) of $p(s)/q(s)$. The Nyquist stability criterion can be formulated as follows ([7], §V.2): Let $q(j\omega) \neq 0$ for $\omega \in [0, \infty)$ and let $q(s)$ have k zeros in the open right half plane. Given a nonzero $\alpha \in \mathbf{R}$, $\phi(s, \alpha) \in \mathcal{H}$ if and only if the magnitude $|p(j\omega)/q(j\omega)|$ is different from zero for all $\omega \in [0, \infty)$ and the net change in the angle of the vector $V(\omega, \alpha)$ pointing from $-\alpha^{-1}$ to a point on the Nyquist plot of $p(s)/q(s)$ as ω increases from 0 to ∞ is equal to $k\pi$. Since $\tilde{G}(\omega)$ is a rational function of ω, the Nyquist plot of $p(s)/q(s)$ has only a finite number of intersections with the real axis. Moreover, by the geometry of the Nyquist plot, the net change in the angle of $V(\omega, \alpha)$ will be the same for all points between any two consecutive intersections. To determine the set $A(p,q)$, it is thus necessary to compute the change in the angle of $V(\omega, \alpha_i)$ only at a finite number of points α_i. A complete Nyquist plot of $p(s)/q(s)$, however, must be drawn. The restrictive assumptions that $q(s)$ has no $j\omega$-axis zeros and that $\alpha \neq 0$ can be removed without difficulty [7]. Moreover, a similar criterion can be stated on the *inverse Nyquist plot* which is the Nyquist plot of $q(s)/p(s)$. We finally note that the logarithmic frequency response graphs *Bode plots* [1] can also be used for a graphical determination of $A(p,q)$.

(iii) *Hurwitz determinants* [6]: Given $q(s) \in \mathbf{R}[s]$ with $deg\, q = n$, let

$$q(s) = a_0 s^n + b_0 s^{n-1} + a_1 s^{n-2} + b_1 s^{n-3} + \ldots \quad (a_0 \neq 0). \tag{3}$$

The Hurwitz matrix of $q(s)$ is ([3], §XV.6) the $n \times n$ matrix

$$\mathcal{H}(q) := \begin{bmatrix} b_0 & b_1 & b_2 & \dots & b_{n-1} \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & b_0 & b_1 & \dots & b_{n-2} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & b_0 & \dots & b_{n-3} \\ .. & .. & .. & \dots & .. \end{bmatrix},$$

where a_k and b_k are zero if they do not appear in (3). Its successive principal minors $\Delta_1 = b_0,\ \Delta_2 = b_0 a_1 \Leftrightarrow a_0 b_1, ..., \Delta_n = det\,\mathcal{H}(q)$ are called the Hurwitz determinants. The Hurwitz criterion for stability is that $q \in \mathcal{H}$ if and only if

$$a_0\Delta_1 > 0,\ \Delta_2 > 0,\ a_0\Delta_3 > 0, ..., a_0^{n\,mod\,2}\Delta_n > 0.$$

For $\phi(s,\alpha) = q(s) + \alpha p(s)$, the entries of the Hurwitz matrix $\mathcal{H}(\phi)$ are linear in α. The Hurwitz criterion applied to $\mathcal{H}(\phi)$ thus yields n inequalities for polynomials in α. The set $A(p,q)$ is simply the intersection of the sets of α satisfying each inequality. Note that determination of $A(p,q)$ requires the determination of the roots of n polynomials. A shortcut is obtained ([7], §V.4) using a consequence of Orlando's formula ([3], §XV.7): If $\phi(s,\alpha)$ has at least one pair of zeros on $j\omega$-axis, then the last Hurwitz determinant $\Delta_n(\alpha)$ associated with $\phi(s,\alpha)$ is zero. To determine $A(p,q)$, it is therefore only necessary to determine the roots in α of $\Delta_n(\alpha)$. This yields at most n points on the real axis and partitions the real axis into at most $n+1$ intervals. In each interval, the sign pattern of the Hurwitz determinants remain the same. Consequently, $A(p,q)$ is the union of those open intervals at one point of which $\phi(s,\alpha)$ is Hurwitz stable. The diagonal terms of a particular triangularization ([3], §XV.6) of the Hurwitz matrix are the terms in the first column of the *Routh array* [12] and the method of *Routh array* is essentially the same as the method of Hurwitz determinants when applied to our problem.

(iv) *Neimark D-decomposition* [8]: Let

$$q(j\omega) = \tilde{h}(\omega) + j\omega\tilde{g}(\omega)\ \ ,\ \ p(j\omega) = \tilde{f}(\omega) + j\omega\tilde{e}(\omega),$$

where $\tilde{h}, \tilde{g}, \tilde{f}, \tilde{e}$ are real and even polynomials of ω. Then, $\phi(j\omega,\alpha) = [\tilde{h}(\omega) + \alpha\tilde{f}(\omega)] + j\omega[\tilde{g}(\omega) + \alpha\tilde{e}(\omega)]$. If $\phi(s,\alpha)$ has a $j\omega$-axis zero, then as α is real, $\tilde{h}(\omega) + \alpha\tilde{f}(\omega) = 0$ and $\tilde{g}(\omega) + \alpha\tilde{e}(\omega) = 0$. Eliminating α from these two equalities, we have

$$\omega[\tilde{g}(\omega)\tilde{f}(\omega) \Leftrightarrow \tilde{h}(\omega)\tilde{e}(\omega)] = 0. \tag{4}$$

Consequently, if $\phi(s,\alpha)$ has a $j\omega$-axis zero, then (4) holds for some $\omega \in [0,\infty)$. Let the roots in $[0,\infty)$ of (4) be $\omega_i,\ \ i = 1, ..., \tilde{k}$ and define

$$\alpha_i = \begin{cases} \Leftrightarrow\frac{\tilde{h}(\omega_i)}{\tilde{f}(\omega_i)} & \text{if } \tilde{f}(\omega_i) \neq 0 \\ \\ \Leftrightarrow\frac{\tilde{g}(\omega_i)}{\tilde{e}(\omega_i)} & \text{if } \omega_i\tilde{e}(\omega_i) \neq 0. \end{cases} \tag{5}$$

If $\tilde{f}(\omega_i) = 0$ and $\omega_i \tilde{e}(\omega_i) = 0$, then let $\alpha_i := \infty$. The values α_i so defined satisfy $\phi(j\omega_i, \alpha_i) = 0$ for $i = 1, ..., \tilde{k}$. We have so far shown that $\phi(s, \alpha)$ has a $j\omega$-axis zero for some α if and only if $\alpha \in \{\alpha_i, \;\; i = 1, ..., \tilde{k}\}$. By the continuity of the roots of $\phi(s, \alpha)$ with respect to α, the following description for $A(p, q)$ is immediate: Let $\{\omega_i\}$ be the roots in $[0, \infty)$ of (4) and let $\{\alpha_i\}$ be as defined in (5). Let the distinct values of $\alpha_i, \;\; i = 1, ..., \tilde{k}$ be ordered as

$$\infty > \alpha_{i_1} > ... > \alpha_{i_k} > -\infty$$

and let $\alpha_{i_0} := \infty$ and $\alpha_{i_{k+1}} := -\infty$ for convenience. Then, for $l = 0, ..., \tilde{k}$ the interval $(\alpha_{i_l}, \alpha_{i_{l+1}})$ is in $A(p, q)$ if and only if at one point α in $(\alpha_{i_l}, \alpha_{i_{l+1}})$ the polynomial $\phi(s, \alpha)$ is Hurwitz stable. Since the union of all candidate intervals cover $\mathbf{R}$, this is a complete description of $A(p, q)$. Thus the method only requires the determination of the roots of (4), α_i, and at most $\tilde{k} + 1$ applications of some stability criterion such as Routh or Hurwitz at one interior point of each interval.

The similarity between the methods **(ii)**-**(iv)** should be clear at this point. By (2), the real axis intersections of the Nyquist plot occur at the points in $\{\omega \in [0, \infty) : \tilde{G}(\omega) = 0\}$ which are among the roots $\{\omega_i, \;\; i = 1, ..., \tilde{k}\}$ of (4) by the second expression below

$$\tilde{H}(\omega) = \frac{\tilde{h}(\omega)\tilde{f}(\omega) + \omega^2 \tilde{g}(\omega)\tilde{e}(\omega)}{\tilde{h}(\omega)^2 + \omega^2 \tilde{g}(\omega)^2}, \;\; \tilde{G}(\omega) = \frac{\omega[\tilde{h}(\omega)\tilde{e}(\omega) - \tilde{g}(\omega)\tilde{f}(\omega)]}{\tilde{h}(\omega)^2 + \omega^2 \tilde{g}(\omega)^2}. \tag{6}$$

Also by these equalities, the values of the real axis intersections can be shown to be $\{\alpha_i^{-1}, \;\; i = 1, ..., \tilde{k}\}$. The method of Nyquist plot for the determination of $A(p, q)$ is thus a particular case of Neimark D-decomposition where the tests of stability in the interior points of the intervals are done through the Nyquist stability criterion. On the other hand, using the properties of Hurwitz determinants it can be shown that $\{\alpha_i, \;\; i = 1, ..., \tilde{k}\} \subset \{\alpha \in \mathbf{R} : \Delta_n(\alpha) = 0\}$. Consequently, the refined method of Hurwitz determinants is essentially the same as the method of Neimark D-decomposition. (The methods **(iii)** and **(iv)** however also extend to the cases where $\phi(s, \alpha)$ is any continuous function of a real *vector* α to yield some geometric criteria for the determination of $A(p, q)$ [7], [8].)

The main contribution of this paper is the derivation of a similar method to **(ii)**-**(iv)** that avoids the tests of stability at the intervals of the real axis. This requirement is replaced by checks of the sign pattern of a rational function at the real nonnegative roots of the polynomial (4). Since (4) is an odd polynomial of ω with degree at most $n + m$, it has at most $\frac{n+m}{2} - 1$ nonnegative roots and a root at $\omega = 0$. Consequently, the method only requires **(i)** the determination of the roots of a polynomial of degree at most $\frac{n+m}{2} - 1$ and **(ii)** a finite number of "rational operations". One consequence of our main result is a condition for $A(p, q)$ to consist of precisely one interval on the real axis. This result is of some interest in the study of convex directions (see [11], [5]).

The paper is organized as follows. In Section 2, we state various elementary facts on polynomials and give an extension of Hermite-Biehler theorem ([3], §XV.14). In Section 3, we state and prove the main results, Theorems 1 and 2. In Section 4, we pursue some implications of the main results in the robust stability analysis. The proof of Lemma 1 is given in the Appendix. The main results in this paper are based on the report [10].

2 Signature of Polynomials

In this section, we give some more terminology and notation, state some elementary facts on polynomials and Hurwitz stable polynomials, and give an extension of the Hermite-Biehler theorem.

Given a set of polynomials $\psi_1, ..., \psi_k \in \mathbf{R}[s]$ not all zero and $k > 1$, their *greatest common divisor* (with highest coefficient 1) is unique and it is denoted by $gcd\,\{\psi_1, ..., \psi_k\}$. If $gcd\,\{\psi_1, ..., \psi_k\} = 1$, then we say $(\psi_1, ..., \psi_k)$ is *coprime*. Let $\mathbf{C}$ denote the set of complex numbers and let $\mathbf{C}_-$, $\mathbf{C}_0$, $\mathbf{C}_+$ denote the points in the open left half, $j\omega$-axis, and the open right half of the complex plane, respectively. Also let $\mathbf{C}_{0+}$ denote the points in the closed right half complex plane. Then, the set $\mathcal{H}$ of Hurwitz stable polynomials are

$$\mathcal{H} = \{\psi(s) \in \mathbf{R}[s] : p(s) = 0 \Rightarrow s \in \mathbf{C}_-\}.$$

The constant nonzero polynomials, i.e., the nonzero elements of $\mathbf{R}$, are thus considered Hurwitz stable. The *signature* $\sigma(\psi)$ *of a polynomial* $\psi \in \mathbf{R}[s]$ is the difference between the number of its $\mathbf{C}_-$ roots and $\mathbf{C}_+$ roots. The signature thus disregards the $j\omega$-axis zeros of the polynomial. Nevertheless, $\psi \in \mathcal{H} \Leftrightarrow \sigma(\psi) = deg\,\psi$ holds.

If $\{r_1, ..., r_t\}$ are a finite number of real numbers and $\mathcal{I}$ is a subset of $\{1, ..., t\}$, then

$$\max_{i \in \mathcal{I}} r_i, \;\; \min_{i \in \mathcal{I}} r_i$$

denote the maximum and the minimum of the numbers r_i as i runs in $\mathcal{I}$. If $\mathcal{I}$ is the empty set, then the maximum is taken as $\Leftrightarrow\infty$ and the minimum is taken as $+\infty$, for convenience. We will also use the notation $r(\pm\infty)$ for the limit as $s \to \pm\infty$ of a real rational function $r(s)$.

Given $\psi \in \mathbf{R}[s]$, *the even-odd components* (a, b) *of* $\psi(s)$ are the unique polynomials $a, b \in \mathbf{R}[u]$ such that $\psi(s) = a(s^2) + sb(s^2)$. The even-odd components of a polynomial and the real and imaginary parts of $\psi(j\omega)$, $\tilde{a}(\omega) := Re\,\{\psi(j\omega)\}$ and $\tilde{b}(\omega) := Im\,\{\psi(j\omega)\}$, are related by

$$\tilde{a}(\omega) = a(\Leftrightarrow\omega^2), \;\; \tilde{b}(\omega) = \omega b(\Leftrightarrow\omega^2).$$

Also note that

$$deg\,\psi \text{ is even } \;\Rightarrow\; \left\{ \begin{array}{l} deg\,a = \frac{deg\,\psi}{2} \\ deg\,b < \frac{deg\,\psi}{2} \end{array} \right\}, \;\; deg\,\psi \text{ is odd } \;\Rightarrow\; \left\{ \begin{array}{l} deg\,a \le \frac{deg\,\psi - 1}{2} \\ deg\,b = \frac{deg\,\psi - 1}{2} \end{array} \right\}. \qquad (7)$$

If $\psi \neq 0$, then $d := gcd\,\{a, b\}$ is well-defined. Since $d(u_0) = 0$ for $u_0 \in \mathbf{C}$ if and only if $s_1 = \sqrt{u_0}$ and $s_2 = \Leftrightarrow\sqrt{u_0}$ are both roots of $\psi(s)$, the roots of $d(s^2)$ correspond to roots of $\psi(s)$ which are symmetrically located with respect to the origin in the complex plane. As a consequence, if $d(u) \neq 0 \;\; \forall u \le 0$, then $\psi(s)$ has no roots on $\mathbf{C}_0$ *except* possibly a simple zero (i.e., a zero of multiplicity 1) at the origin. Also note that if $\psi(s) \in \mathcal{H}$, then $d = 1$ since otherwise there would be at least one root of $\psi(s)$ in $\mathbf{C}_{0+}$. It is actually possible to state a necessary and sufficient condition for the Hurwitz stability of ψ in terms of its even-odd components (a, b). This result is known as the Hermite-Biehler theorem. We state it in a suitable form for our purpose. Let us define the *signum function* $\mathcal{S} : \mathbf{R} \to \{-1, 0, 1\}$ by

$$\mathcal{S}r = \begin{cases} -1 & \text{if } r < 0 \\ 0 & \text{if } r = 0 \\ 1 & \text{if } r > 0. \end{cases}$$

Proposition 1 ([3], §XV, 14) *A nonzero polynomial $\psi \in \mathbf{R}[s]$ is Hurwitz stable if and only if its even-odd components (a, b) are such that $b \not\equiv 0$ and at the distinct real negative roots $v_1 > v_2 > ... > v_k$ of b the following holds:*

$$deg\,\psi = \begin{cases} \mathcal{S}b(0)[\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + ... + (-1)^k 2\mathcal{S}a(v_k)] & \textit{for } deg\,\psi \textit{ odd} \\ \mathcal{S}b(0)[\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + ... + (-1)^k 2\mathcal{S}a(v_k) + (-1)^{k+1}\mathcal{S}a(-\infty)] & \textit{for } deg\,\psi \textit{ even.} \end{cases} \quad (8)$$

By (7), if $deg\,\psi$ is odd, then $deg\,b = (deg\,\psi - 1)/2$ so that $deg\,\psi \geq 2k + 1$. However, the maximum value the right hand side of (8) can attain is also $2k + 1$. Similarly, if $deg\,\psi$ is even, then it is easy to see by (7) that $deg\,\psi \geq 2k + 2$ which is the maximum value the right hand side of (8) can attain. It follows that (8) is satisfied if only if $k = deg\,b$, $\mathcal{S}a(0) = \mathcal{S}b(0)$, and in each interval $(v_1, 0)$, (v_2, v_1), ..., the polynomial a has exactly one root. Such an (a, b) is called a *positive pair* ([3], §XV, 14) and the proposition reads: $\psi \in \mathcal{H}$ if and only if (a, b) is a positive pair. The following is a generalization of Proposition 1 to not necessarily Hurwitz stable polynomials.

Lemma 1. *Let a nonzero polynomial $\psi \in \mathbf{R}[s]$ have the even-odd components (a, b). Suppose $b \not\equiv 0$ and (a, b) is coprime. Then, $\sigma(\psi) = r$ if and only if at the real negative roots of odd multiplicities $v_1 > v_2 > ... > v_k$ of b the following holds:*

$$r = \begin{cases} \mathcal{S}b(0_-)[\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + ... + (-1)^k 2\mathcal{S}a(v_k)] & \textit{for } deg\,\psi \textit{ odd} \\ \mathcal{S}b(0_-)[\mathcal{S}a(0) - 2\mathcal{S}a(v_1) + ... + (-1)^k 2\mathcal{S}a(v_k) + (-1)^{k+1}\mathcal{S}a(-\infty)] & \textit{for } deg\,\psi \textit{ even,} \end{cases} \quad (9)$$

where $b(0_-) := (-1)^{m_0} b^{(m_0)}(0)$, m_0 is the multiplicity of $u = 0$ as a root of $b(u)$, and $b^{(m_0)}(0)$ denotes the value at $u = 0$ of the m_0-th derivative of $b(u)$.

Proof. See the Appendix.[3] □

3 The Set of Stabilizing Gains

We now return to our problem. Let $p, q \in \mathbf{R}[s]$ be nonzero, with $m = deg\,p$ and $n = deg\,q$ and satisfy

(A1) $n \geq m, \quad n \geq 1.$

[3] It would be surprising if this result is not already known in some form or other. However, we have not been able to locate an appropriate reference and a proof is supplied.

(A2) (p, q) is coprime.

In this section we obtain analytic descriptions of $A(p,q)$ through two closely related procedures in Theorems 1 and 2 under assumptions (A1) and (A2). Note that if (A1) fails, then either $n < m$ in which case $A(p,q) = \emptyset$ or $n = 0$ in which case $A(p,q) = \mathbf{R} \setminus \{\Leftrightarrow q/p\}$. On the other hand, if (A2) fails, then with $\tilde{\phi} := gcd\{p, q\}$, we have $q = \tilde{\phi}\tilde{q}$ and $p = \tilde{\phi}\tilde{p}$ for coprime polynomials $(\tilde{q}, \tilde{p})$. Then, $A(p,q) \neq \emptyset$ if and only if $\tilde{\phi} \in \mathcal{H}$ and $A(\tilde{p}, \tilde{q}) \neq \emptyset$, in which case $A(p,q) = A(\tilde{p}, \tilde{q})$. Consequently, we can assume (A1) and (A2) without loss of generality.

Let (h, g) and (f, e) be the even-odd components of q and p, respectively, so that $q(s) = h(s^2) + sg(s^2)$, $p(s) = f(s^2) + se(s^2)$. By (A1), f and e are not both zero and $d := gcd\{f, e\}$ is well-defined. Let

$$f = d\bar{f}, \;\; e = d\bar{e}$$

for coprime polynomials $\bar{f}$, $\bar{e} \in \mathbf{R}[u]$. Then, the polynomial

$$\bar{p}(s) := \bar{f}(s^2) + s\bar{e}(s^2) = p(s)/d(s^2) \tag{10}$$

is free of $\mathbf{C}_0$ roots except possibly a simple root at $s = 0$. Let (H, G) be the even-odd components of $q(s)\bar{p}(\Leftrightarrow s)$. Also let $F(s^2) := p(s)\bar{p}(\Leftrightarrow s)$. By a simple computation, it follows that

$$\begin{array}{rcl} H(u) & = & h(u)\bar{f}(u) \Leftrightarrow ug(u)\bar{e}(u), \\ G(u) & = & g(u)\bar{f}(u) \Leftrightarrow h(u)\bar{e}(u), \\ F(u) & = & f(u)\bar{f}(u) \Leftrightarrow ue(u)\bar{e}(u). \end{array} \tag{11}$$

These polynomials are related to $q(j\omega)/p(j\omega)$ by

$$\frac{H}{F}(\Leftrightarrow \omega^2) = Re\{\frac{q(j\omega)}{p(j\omega)}\}, \;\; \Leftrightarrow \omega\frac{G}{F}(\Leftrightarrow \omega^2) = Im\{\frac{q(j\omega)}{p(j\omega)}\}$$

whenever defined. If $G \not\equiv 0$ and if they exist, let the *real negative zeros with odd multiplicities of* $G(u)$ be $\{v_1, ..., v_k\}$ with the ordering

$$v_1 > v_2 > \cdots > v_k, \tag{12}$$

with $v_0 := 0$ and $v_{k+1} := \Leftrightarrow \infty$ for notational convenience, and let the *real negative zeros with even multiplicities of* $G(u)$ be $\{u_1, ..., u_l\}$.

Lemma 2. *Given* $p, q \in \mathbf{R}[s]$ *satisfying (A1), (A2), let* F, G, H *be defined by (11). A real number* α *is in* $A(p,q)$ *if and only if* $G \not\equiv 0$, $(H + \alpha F, G)$ *is coprime, and* $\sigma[\psi(s,\alpha)] = n \Leftrightarrow \sigma[\bar{p}(s)]$, *where* $\psi(s, \alpha) := H(s^2) + \alpha F(s^2) + sG(s^2)$.

Proof. Note that by (11), $\psi(s,\alpha) = \phi(s,\alpha)\bar{p}(\Leftrightarrow s)$ and that s_0 is a root of $\bar{p}(\Leftrightarrow s)$ if and only if $\Leftrightarrow s_0$ is a root of $\bar{p}(s)$. If $\alpha \in A(p,q)$, then $\sigma(\phi) = n$ and $\sigma(\psi) = n \Leftrightarrow \sigma(\bar{p})$. Suppose $gcd\{H + \alpha F, G\} \neq 1$. Since $(H + \alpha F, G)$ are the even-odd components of $\psi(s, \alpha)$, it follows that $s_0 = \mp\sqrt{u_0}$ are both roots of $\psi(s,\alpha)$ for some root $u_0 \in \mathbf{C}$ of $gcd\{H+\alpha F, G\}(u)$. If $Re\{s_0\} = 0$, then as $\phi(s,\alpha)$ is Hurwitz stable both should be roots of $\bar{p}(\Leftrightarrow s)$. This is not possible since $\bar{p}(s)$ has no zeros in $\mathbf{C}_0$ except possibly a simple zero at $s=0$. Hence $Re\{s_0\} \neq 0$ and one of the roots, say $s_0 = \Leftrightarrow\sqrt{u_0}$, is in $\mathbf{C}_+$. Since ϕ is Hurwitz stable, s_0 is a root of $\bar{p}(\Leftrightarrow s)$. Since $gcd\,(\bar{f},\bar{e}) = 1$, $\Leftrightarrow s_0$ can not also be a root of $\bar{p}(\Leftrightarrow s)$ so that it is a root of $\phi(s,\alpha)$. But $\phi(\Leftrightarrow s_0,\alpha) = q(\Leftrightarrow s_0) + \alpha p(\Leftrightarrow s_0) = 0$ implies by $\bar{p}(\Leftrightarrow s_0) = 0$ that $q(\Leftrightarrow s_0) = 0$. This contradicts the assumption (A2). Now if $G \equiv 0$, then by coprimeness of $(H+\alpha F, G)$, $\psi(s,\alpha)$ is a constant. This implies that $n=0$ which contradicts the assumption (A1). Conversely, suppose $G \not\equiv 0$ and for some $\alpha \in \mathbf{R}$, $(H+\alpha F, G)$ is coprime and $\sigma(\psi) = n \Leftrightarrow \sigma(\bar{p})$. Hence $\sigma(\phi) = n$ and all roots of ϕ are in $\mathbf{C}_-$. □

Theorem 1. *Let $p, q \in \mathbf{R}[s]$ satisfy the assumptions (A1), (A2) and let F, G, H, $\{v_i\}$ be defined by (11), (12).*

[Existence] *The set $A(p,q)$ is nonempty if and only if*

(i) $G \not\equiv 0$,

(ii) (F,G,H) *is coprime,*

(iii) There exists a sequence of signums

$$\mathcal{I} = \begin{cases} \{i_0, i_1, \ldots, i_k\} & \text{for odd } n \Leftrightarrow m \\ \{i_0, i_1, \ldots, i_{k+1}\} & \text{for even } n \Leftrightarrow m, \end{cases}$$

where $i_0 \in \{\Leftrightarrow 1, 0, 1\}$ and $i_j \in \{\Leftrightarrow 1, 1\}$ for $j = 1, \ldots, k+1$ satisfying (1)-(3):

(1) $$F(v_j) = 0 \quad \Rightarrow \quad i_j = \mathcal{S}H(v_j)\mathcal{S}G(0_-), \;\; j = 0, 1, ..., k.$$

(2) $$n \Leftrightarrow \sigma(p) = \begin{cases} i_0 \Leftrightarrow 2i_1 + 2i_2 + \cdots + 2(\Leftrightarrow 1)^k i_k & \text{for odd } n \Leftrightarrow m \\ i_0 \Leftrightarrow 2i_1 + 2i_2 + \cdots + 2(\Leftrightarrow 1)^k i_k + (\Leftrightarrow 1)^{k+1} i_{k+1} & \text{for even } n \Leftrightarrow m. \end{cases}$$

(3) $$\max_{j\in\mathcal{J}^-} \frac{H}{F}(v_j) \; < \; \min_{j\in\mathcal{J}^+} \frac{H}{F}(v_j) \quad \text{if} \quad G(0_-) > 0,$$

$$\max_{j\in\mathcal{J}^+} \frac{H}{F}(v_j) \; < \; \min_{j\in\mathcal{J}^-} \frac{H}{F}(v_j) \quad \text{if} \quad G(0_-) < 0,$$

where $\mathcal{J}^+ := \{j : i_j \in \mathcal{I},\; i_j\mathcal{S}F(v_j) = 1\}$ and $\mathcal{J}^- := \{j : i_j \in \mathcal{I},\; i_j\mathcal{S}F(v_j) = \Leftrightarrow 1\}$ and where $G(0_-) := (\Leftrightarrow 1)^{m_0} G^{(m_0)}(0)$ with m_0 being the multiplicity of $u=0$ as a root of $G(u)$.

[Determination] *Let (i)-(iii) hold. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_\mu$ be the set of all signum sequences that satisfy (iii) and let $\mathcal{J}_t^{\pm} := \{j : i_j \in \mathcal{I}_t,\; i_j\mathcal{S}F(v_j) = \pm 1\}$ for $t = 1, ..., \mu$. Consider the μ*

open intervals defined by

$$A_t := \begin{cases} (\Leftrightarrow \min\limits_{j \in \mathcal{J}_t^+} \dfrac{H}{F}(v_j), \ \ \Leftrightarrow \max\limits_{j \in \mathcal{J}_t^-} \dfrac{H}{F}(v_j)) & \textit{if} \ \ G(0_-) > 0 \\ \\ (\Leftrightarrow \min\limits_{j \in \mathcal{J}_t^-} \dfrac{H}{F}(v_j), \ \Leftrightarrow \max\limits_{j \in \mathcal{J}_t^+} \dfrac{H}{F}(v_j)) & \textit{if} \ \ G(0_-) < 0 \end{cases} \tag{13}$$

for $t = 1, 2, \cdots, \mu$ *and the set of points*

$$\hat{A} := \begin{cases} \{\Leftrightarrow \frac{H}{F}(u_j) : F(u_j) \neq 0\} & \textit{if} \ \ n > m \\ \\ \{\Leftrightarrow \frac{H}{F}(u_j) : F(u_j) \neq 0\} \cup \{\Leftrightarrow \frac{q}{p}(\infty)\} & \textit{if} \ \ n = m. \end{cases}$$

Then,

$$A(p,q) = \bigcup_{t=1}^{\mu} A_t \setminus (\hat{A} \cap A_t). \tag{14}$$

Proof. (Only If) Let $A(p,q) \neq \emptyset$ and let $\alpha \in A(p,q)$. Thus, $\phi(s,\alpha) \in \mathcal{H}$ and, by Lemma 2, $G \not\equiv 0$, $(H + \alpha F, G)$ is coprime (implying that (F, G, H) is also coprime), and $\sigma(\psi) = n \Leftrightarrow \sigma(\bar{p})$, where $\psi(s,\alpha) = \phi(s,a)\bar{p}(\Leftrightarrow s)$. Since $(H + \alpha F, G)$ are even-odd components of ψ and since $deg\, \psi = n + deg\, \bar{p}$ is odd if and only if $n \Leftrightarrow m$ is odd, it follows by Lemma 1 that at the roots v_j of $G(u)$, (9) holds with $a(u) := H(u) + \alpha F(u)$ and $b(u) := G(u)$. Therefore, the sequence of signums $\mathcal{I} = \{i_j\}$ defined by

$$i_j = \begin{cases} \mathcal{S}(H + \alpha\, F)(v_j) & \text{if } G(0_-) > 0 \\ \Leftrightarrow \mathcal{S}(H + \alpha\, F)(v_j) & \text{if } G(0_-) < 0 \end{cases} \tag{15}$$

for $j = 0, 1, \ldots, k, k+1$ satisfies (1) and (2) of (iii). Note that, by (ii), $i_j \neq 0$ except when $\bar{p}(0) = 0$ so that $i_j \in \{\Leftrightarrow 1, 1\}$ for $j = 1, ..., k+1$ and $i_0 \in \{\Leftrightarrow 1, 0, 1\}$, where $i_0 = 0$ if and only if $\bar{p}(0) = 0$. To prove that α satisfies (3) of (iii), let us first suppose $G(0_-) > 0$. Then, by (15), we get

$$\alpha > \Leftrightarrow \frac{H}{F}(v_j) \ \ \text{for all} \ \ v_j \ \ \text{for which} \ \ i_j \mathcal{S} F(v_j) = 1,$$

$$\alpha < \Leftrightarrow \frac{H}{F}(v_j) \ \ \text{for all} \ \ v_j \ \ \text{for which} \ \ i_j \mathcal{S} F(v_j) = \Leftrightarrow 1,$$

where for $j = k+1$ the fact that " $deg\, H \geq deg\, F$ for even $n \Leftrightarrow m$" is used, see (7). It follows that

$$\max_{\{j \,:\, i_j \mathcal{S} F(v_j) = 1\}} \Leftrightarrow \frac{H}{F}(v_j) \ \ < \ \ \alpha \ \ < \ \ \min_{\{j \,:\, i_j \mathcal{S} F(v_j) = -1\}} \Leftrightarrow \frac{H}{F}(v_j),$$

or equivalently,

$$\Leftrightarrow \min_{\{j\,:\,i_j\mathcal{S}F(v_j)=1\}} \frac{H}{F}(v_j) \;<\; \alpha \;<\; \Leftrightarrow \max_{\{j\,:\,i_j\mathcal{S}F(v_j)=-1\}} \frac{H}{F}(v_j).$$

This yields the first inequality in (3). The second inequality in (3) is shown similarly in the case $G(0_-) < 0$. This proves the "only if" part of the "existence" statement. By coprimeness of $(H + \alpha F, G)$ and by $deg\,\phi(s, \alpha) = deg\,q$, we have $\alpha \notin \hat{A}$. Therefore, by (3), $A(p, q) \subset A$, where A denotes the right hand side of (14).

(If) Suppose (i)-(iii) are satisfied. We prove that $A \subset A(p, q)$ establishing the "if" part of the "existence" statement as well as the description for $A(p, q)$. Let us first consider

$$A_c := A \cap \{\alpha \in \mathbf{R} : (H + \alpha F, G) \text{ is coprime}\}.$$

By the definition of the set A_c, $(H + \alpha F, G)$ is coprime for all $\alpha \in A_c$ and, by (i), $G \not\equiv 0$. Let $\alpha \in A_c$ belong to the interval A_ν obtained by a signum set $\mathcal{I}_\nu$ for some $\nu \in \{1, ..., \mu\}$. Thus, as (3) holds for $\mathcal{J}_\nu^-$ and $\mathcal{J}_\nu^+$, we have $\pm\mathcal{S}(H + \alpha F)(v_j) = i_j$ for $\mathcal{S}G(0_-) = \pm$ for all $i_j \in \mathcal{I}_\nu$. By (2) of (iii), it follows that $a := H + \alpha F$, $b := G$ satisfy (9) of Lemma 1 so that $\sigma(\phi(s, \alpha)\bar{p}(\Leftrightarrow s)) = n \Leftrightarrow \sigma(\bar{p}(s))$. By Lemma 2, it follows that $A_c \subset A(p, q)$. We now show that the set $A \setminus A_c$ of *finite number of points* is empty. Suppose $\alpha_0 \in A \setminus A_c$ so that there exists $u_0 \in \mathbf{C}$ satisfying $H(u_0) + \alpha_0 F(u_0) = 0$, $G(u_0) = 0$. If $F(u_0) = 0$, then $gcd\,\{F, G, H\} \neq 0$ which contradicts (ii). Thus, $F(u_0) \neq 0$. We consider two cases. First, suppose u_0 is real and nonpositive. Then, $u_0 \in \{v_0, ..., v_k, u_1, ..., u_l\}$ and $\alpha_0 = \Leftrightarrow H(u_0)/F(u_0)$. This contradicts the fact that $\alpha_0 \in A$. Second, suppose that u_0 is either a real positive number or a nonreal complex number. It follows that $\phi(\pm\sqrt{u_0}, \alpha_0)\bar{p}(\mp\sqrt{u_0}) = 0$ since u_0 is a common zero of the even-odd components of $\phi(s, \alpha_0)\bar{p}(\Leftrightarrow s)$. Note that both $\pm\sqrt{u_0}$ can not be roots of $\bar{p}(s)$ since the latter has coprime even-odd components. On the other hand, if $\bar{p}(\pm\sqrt{u_0}) = 0$ and $\phi(\mp\sqrt{u_0}) = 0$, then (p, q) is not coprime and (A2) is contradicted. Hence, both of $\pm\sqrt{u_0}$ are the roots of $\phi(s, \alpha)$. Note that $Re\{\sqrt{u_0}\} \neq 0$ as u_0 is either real positive or nonreal complex. Consequently, $\phi(s, \alpha)$ has a root in $\mathbf{C}_+$. But, since A_c is dense in A, there exists $\alpha_1 \in A_c$ arbitrarily close to α_0 for which $\phi(s, \alpha_1)$ is Hurwitz stable. By the continuity of the roots of ϕ with respect to α and by the fact that $\mathbf{C}_- \cap \mathbf{C}_+ = \emptyset$, such an α_0 can not exist. We have thus shown that $A \setminus A_c$ is empty and hence $A \subset A(p, q)$. □

Remarks. 1. By condition (2) of (iii), some of the elements of $\mathcal{I}$ may be fixed. If $\bar{p}(0) \neq 0$, then the fixed elements are determined by v_j for which $F(v_j) = 0$ for some $j = 0, 1, ..., k$. Since $F(u) = d(u)\bar{F}(u)$ where $\bar{F}(s^2) := \bar{p}(s)\bar{p}(\Leftrightarrow s)$, we have $\bar{F}(u) > 0$ for all $u \leq 0$ and the roots of G which yield fixed elements are among the real negative roots of $gcd\{G, d\}$. On the other hand, if $\bar{p}(0) = 0$, then $H(0) = 0$, $F(0) = 0$ which fixes $i_0 = 0$. The real negative roots of $d(u)$ yield pairs of zeros of $p(s)$ in $\mathbf{C}_0$. By these considerations, it is easy to see that, the fixed signums in $\mathcal{I}$ occur if and only if either $p(s)$ has a zero $j\omega \neq 0$ such that $u = \Leftrightarrow\omega^2$ is a zero of $G(u)$ or $\bar{p}(0) = 0$.

2. Suppose $p(s)$ has no roots in $\mathbf{C}_0$ and let $n \Leftrightarrow m$ be even. Then, there are no fixed signums in $\mathcal{I}$ by Remark 1. In this case, there are 2^{k+2} different candidate signum sequences to satisfy (2) and (3) in Theorem 1. With $l := n - \sigma(p)$, it is easy to compute that among these

$$\frac{(k+1)!}{[(2k+2-l)/4]![(2k+2+l)/4]!} \quad \text{if } (l+2k)\, mod\, 4 = 2$$

$$2\,\frac{k!}{[(2k-l)/4]![(2k+l)/4]!} \quad \text{if } (l+2k)\, mod\, 4 = 0$$

different signum sequences satisfy (2) and are candidate sequences to satisfy condition (3) in Theorem 1.

3. Two different signum sequences $\mathcal{I}_1, \mathcal{I}_2$ satisfying (iii) yield two disjoint intervals A_1, A_2. To see this, suppose $\alpha \in A_1 \cap A_2$. Then, by the "only if" part of the proof of Theorem 1, (15) holds for signums of both $\mathcal{I}_1$ and $\mathcal{I}_2$ and they are identical. Consequently, there may be at most $k+2$ different signum sequences that satisfy (iii) in Theorem 1. $\triangle$

Let us now consider the set

$$B(p,q) := \{\beta \in \mathbf{R} : \theta(s,\beta) = \beta q(s) + p(s) \in \mathcal{H} \ , \ \ deg\, \theta = deg\, q\}.$$

If (A1) and (A2) hold, then the following relation between $A(p,q)$ and $B(p,q)$ is immediate. If $\alpha \in A(p,q)$ and $\alpha \neq 0$, then $\beta := \alpha^{-1}$ is in $B(p,q)$. If $0 \in A(p,q)$, then $q \in \mathcal{H}$ and the intervals (β_1, ∞), $(-\infty, -\beta_2)$ are contained in $B(p,q)$ for some $\beta_1,\ \beta_2 > 0$. If $\beta \in B(p,q)$ and $\beta \neq 0$, then $\alpha := \beta^{-1}$ is in $A(p,q)$. If $0 \in B(p,q)$, then $n = m$, $p \in \mathcal{H}$, and the intervals (α_1, ∞), $(-\infty, -\alpha_2)$ are contained in $A(p,q)$ for some $\alpha_1,\ \alpha_2 > 0$.

We now state a counterpart to Theorem 1 which states conditions for $B(p,q)$ to be nonempty and gives a description of $B(p,q)$.

By (A1), h and g are not both zero and $b := gcd\,\{h, g\}$ is well-defined. Let

$$h = b\bar{h}, \ \ g = b\bar{g}$$

for coprime polynomials $\bar{h}$, $\bar{g} \in \mathbf{R}[u]$. Then, the polynomial

$$\bar{q}(s) := \bar{h}(s^2) + s\bar{e}(s^2) = q(s)/b(s^2) \tag{16}$$

is free of $\mathbf{C}_0$ roots except possibly a simple root at $s = 0$. Let (E, D) be the even-odd components of $p(s)\bar{q}(-s)$ and let $C(s^2) := \bar{q}(s)\bar{q}(-s)$. Similar to (11), we have

$$\begin{aligned} E(u) &= \bar{h}(u)f(u) - u\bar{g}(u)e(u), \\ D(u) &= \bar{h}(u)e(u) - \bar{g}(u)f(u), \\ C(u) &= \bar{h}(u)h(u) - u\bar{g}(u)g(u). \end{aligned} \tag{17}$$

By (2) and (6), we have

$$\frac{E}{C}(-\omega^2) = Re\{\frac{p(j\omega)}{q(j\omega)}\}, \ \ \omega\frac{D}{C}(-\omega^2) = Im\{\frac{p(j\omega)}{q(j\omega)}\}$$

whenever defined. If $D \not\equiv 0$ and if they exist, let the *real negative zeros with odd multiplicities of* $D(u)$ be $\{x_1, ..., x_k\}$ with the ordering

$$x_1 > x_2 > \cdots > x_k, \tag{18}$$

with $x_0 := 0$ and $x_{k+1} := -\infty$ for notational convenience, and let the *real negative zeros with even multiplicities of* $D(u)$ be $\{y_1, ..., y_l\}$.[4]

Theorem 2. *Let $p, q \in \mathbf{R}[s]$ satisfy the assumptions (A1), (A2) and let C, D, E, $\{x_j\}$ be defined by (17), (18).*

[**Existence**] *The set $B(p, q)$ is nonempty if and only if*

(i) $D \not\equiv 0$,

(ii) (C, D, E) *is coprime,*

(iii) There exists a sequence of signums

$$\mathcal{I} = \{i_0, i_1, \ldots, i_{k+1}\}$$

where $i_0 \in \{-1, 0, 1\}$ and $i_j \in \{-1, 1\}$ for $j = 1, \ldots, k+1$ satisfying (1)-(3):

(1) $$C(x_j) = 0 \;\;\Rightarrow\;\; i_j = \mathcal{S}E(x_j)\mathcal{S}D(0_-), \;\; j = 0, 1, ..., k.$$

(2) $$n - \sigma(q) = i_0 - 2i_1 + 2i_2 + \cdots + 2(-1)^k i_k + (-1)^{k+1} i_{k+1}.$$

(3) $$\begin{aligned} \max_{j \in \mathcal{J}^-} \frac{E}{C}(x_j) \;&<\; \min_{j \in \mathcal{J}^+} \frac{E}{C}(x_j) \quad \text{if} \quad D(0_-) > 0, \\ \max_{j \in \mathcal{J}^+} \frac{E}{C}(x_j) \;&<\; \min_{j \in \mathcal{J}^-} \frac{E}{C}(x_j) \quad \text{if} \quad D(0_-) < 0, \end{aligned}$$

where $\mathcal{J}^+ := \{j : i_j \in \mathcal{I},\; i_j\mathcal{S}C(x_j) = 1\}$ and $\mathcal{J}^- := \{j : i_j \in \mathcal{I},\; i_j\mathcal{S}C(x_j) = -1\}$ and where $D(0_-) := (-1)^{n_0} D^{(n_0)}(0)$ with n_0 being the multiplicity of $u = 0$ as a root of $D(u)$.

[**Determination**] *Let (i)-(iii) hold. Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_\mu$ be the set of all signum sequences that satisfy (iii) and let $\mathcal{J}_t^\pm := \{j : i_j \in \mathcal{I}_t,\; i_j\mathcal{S}C(v_j) = \pm 1\}$ for $t = 1, ..., \mu$. Consider μ open intervals defined by*

$$B_t := \begin{cases} \left(-\min\limits_{j \in \mathcal{J}_t^+} \dfrac{E}{C}(x_j),\;\; -\max\limits_{j \in \mathcal{J}_t^-} \dfrac{E}{C}(x_j)\right) & \text{if} \quad D(0_-) > 0 \\[2ex] \left(-\min\limits_{j \in \mathcal{J}_t^-} \dfrac{E}{C}(x_j),\;\; -\max\limits_{j \in \mathcal{J}_t^+} \dfrac{E}{C}(x_j)\right) & \text{if} \quad D(0_-) < 0 \end{cases}$$

[4] There is a slight ambiguity of notation here; D and G may not have the same number of real negative roots of odd (even) multiplicity unless d and b have.

for $t = 1, 2, \cdots, \mu$ *and the set of points*

$$\hat{B} := \begin{cases} \{\Leftrightarrow\frac{E}{C}(y_j) : C(y_j) \neq 0\} \cup \{0\} & \text{if } \ n > m \\ \\ \{\Leftrightarrow\frac{E}{C}(y_j) : C(y_j) \neq 0\} \cup \{\Leftrightarrow\frac{q}{p}(\infty)\} & \text{if } \ n = m \ . \end{cases}$$

Then,

$$B(p, q) = \bigcup_{t=1}^{\mu} B_t \setminus (\hat{B} \cap B_t). \tag{19}$$

Proof. The proof is analogous to the proof of Theorem 1. We only give an outline. Similar to Lemma 2, given $p, q \in \mathbf{R}[s]$ satisfying (A1), (A2), $\beta \in B(p,q)$ if and only if $D \not\equiv 0$, $(E + \beta C, D)$ is coprime, and $\sigma[\psi(s,\beta)] = n \Leftrightarrow \sigma[\bar{q}(s)]$, where $\psi(s,\beta) := E(s^2) + \beta C(s^2) + sD(s^2)$. Applying Lemma 1 with $a := E + \beta C, \ \ b := D$ and noting that $deg\, \psi = deg\, \theta + deg\, \bar{q}$ is even, it follows that the signum sequence $\bar{\mathcal{I}} = \{\bar{i}_j\}$, where

$$\bar{i}_j = \begin{cases} \mathcal{S}(E + \beta C)(x_j) & \text{if } D(0_-) > 0 \\ \Leftrightarrow\mathcal{S}(E + \beta C)(x_j) & \text{if } D(0_-) < 0, \end{cases} \tag{20}$$

satisfies (iii) provided $\beta \in B(p,q)$. Conversely, if a signum sequence $\mathcal{I}$ satisfying (iii) exists, then again by Lemma 1, the polynomial $\psi(s,\beta)$ will have the signature $n \Leftrightarrow \sigma(q)$ for any $\beta \in B$, where B is the right hand side of (19), so that $\beta \in B \Rightarrow \beta \in B(p,q)$. □

Remark 4. Remarks (1)-(3) apply to Theorem 2 with appropriate modifications. The fixed signums in $\mathcal{I}$ occur if and only if either $q(s)$ has a zero $j\omega \neq 0$ such that $u = \Leftrightarrow\omega^2$ is a zero of $D(u)$ or $\bar{q}(0) = 0$. If there are no fixed signums in $\mathcal{I}$, then the number of different signum sequences satisfying (2) of Theorem 2 is again approximately the number given in Remark 2. Finally, two different signum sequences satisfying (iii) yield disjoint open intervals all (except finitely many) points of which are in $B(p,q)$. △

Remark 5. Theorem 2 is an analytic version of the Nyquist stability criterion outlined in Section 1, (ii). Similarly, Theorem 1 is an analytic version of the inverse Nyquist criterion. We now give an explicit connection between the signum sequences in Theorems 1 and 2 and the intervals they yield under the assumptions (A1), (A2), and

(A3) $b = gcd\,\{h, g\} = 1, \ \ d = gcd\,\{f, e\} = 1.$

By (A3), $H = E$, $G = \Leftrightarrow D$, $G(0_-) = \Leftrightarrow D(0_-)$ and $v_j = x_j$ for $j = 0, 1, ..., k+1$. Moreover, for any $\alpha, \beta \in \mathbf{R}$, we have

$$\begin{aligned} H(H + \alpha F) \Leftrightarrow F(\alpha E + C) &= \Leftrightarrow uGD, \\ H(E + \beta C) \Leftrightarrow C(\beta H + F) &= \Leftrightarrow uDG. \end{aligned} \tag{21}$$

Now let $\mathcal{I}_t = \{i_j\}$ satisfy the conditions (1)-(3) of Theorem 1 and yield $A_t = (\alpha_1, \alpha_2)$, all (except possibly finitely many) points of which are in $A(p,q)$. When $n \Leftrightarrow m$ is odd, let $i_{k+1} := \mathcal{S}H(v_{k+1})\mathcal{S}G(0_-)$ for convenience and consider $\bar{\mathcal{I}}_t = \{\bar{i}_j\}$ defined by

$$\bar{i}_j := \begin{cases} \Leftrightarrow i_j \mathcal{S}H(v_j) & \text{if } \ 0 < \alpha_1 < \alpha_2 \\ i_j \mathcal{S}H(v_j) & \text{if } \ \alpha_1 < \alpha_2 < 0 \end{cases} \tag{22}$$

for $j = 0, ..., k, k+1$ if $\bar{p}(0) \neq 0$. If $\bar{p}(0) = 0$, then we let $\bar{i}_j$ be defined by (22) for $j = 1, ..., k+1$ and by

$$\bar{i}_0 = \begin{cases} \mathcal{S}D(0_-) & \text{if} \quad 0 < \alpha_1 < \alpha_2 \\ -\mathcal{S}D(0_-) & \text{if} \quad \alpha_1 < \alpha_2 < 0 \end{cases}.$$

Note that as the interval A_t in the above two cases do not contain the point $\alpha = 0$, by the first equality in (21), the number $\beta = 1/\alpha$ is such that (20) holds. By the "only if" part of the proof of Theorem 2, the signum sequence $\bar{\mathcal{I}}_t$ defined above yields the interval $B_t = (1/\alpha_2, 1/\alpha_1)$, all (except finitely many) points of which are in $B(p,q)$. If $\alpha_1 < 0 < \alpha_2$, then $0 \in A(p,q)$ and the signum sequence $\mathcal{I}_t = \{i_j = \mathcal{S}H(v_j)\}$ satisfies (iii) of Theorem 1 yielding A_t. The constant signum sequences $\{-1\}$ and $\{+1\}$ both satisfy (20) for $\beta \to \pm\infty$, by the second equality in (21) and yield the intervals $B_{t1} = (-\infty, 1/\alpha_1)$ and $B_{t2} = (1/\alpha_2, \infty)$. We note that, given an interval B_t obtained via the signum sequence $\bar{\mathcal{I}}_t$, the procedure of defining $\mathcal{I}_t$ satisfying (iii) of Theorem 1 and yielding A_t is similar and follows by the equalities (21). Finally, the restrictive assumption (A3) can be removed at the expense of a much more detailed analysis. $\triangle$

Example 1. Consider

$$q = s^6 + 2s^5 + 5s^4 + 5s^3 + s^2 + 0.5s - 0.05,$$
$$p = s^6 + 4s^5 + 30s^4 + 60s^3 + 150s^2 + 100s + 100.$$

To determine $A(p,q)$, we first employ Theorem 1. By the method of Hurwitz determinants, it is easy to see that p is Hurwitz stable, i.e., $\sigma(p) = 6$ which also implies that $b = 1$. Using (11), we have

$$F(u) = u^6 + 44u^5 + 720u^4 + 4800u^3 + 16500u^2 + 20000u + 10000,$$
$$G(u) = -2u^5 - 15u^4 + 46.5u^3 + 405.2u^2 + 478u + 55,$$
$$H(u) = u^6 + 27u^5 + 161u^4 + 377.95u^3 + 118.5u^2 + 42.5u - 5.$$

The polynomial $G(u)$ has one positive and four negative real zeros which are

$$v_1 = -0.1289, \; v_2 = -1.3783, \; v_3 = -3.7921, \; v_4 = -7.5823.$$

Since $n - m = 0$ is even and $n - \sigma(p) = 0$, by Remark 2, there are 12 candidate signum sequences $\{i_0, i_1, i_2, i_3, i_4, i_5\}$ that satisfy the condition (2) of item (iii) in Theorem 1. Now, $G(0_-) = G(0) = 55 > 0$, $F(v_i) > 0$ for $i = 0, ..., 5$, and

$$\tfrac{H}{F}(v_0) = -0.0005, \; \tfrac{H}{F}(v_1) = -0.0012, \; \tfrac{H}{F}(v_2) = -0.1041,$$

$$\tfrac{H}{F}(v_3) = -0.1471, \; \tfrac{H}{F}(v_4) = -0.6207, \; \tfrac{H}{F}(v_5) = 1.$$

The signum sequences

$$\mathcal{I}_1 = \{1, 1, 1, 1, 1, 1\}, \; \mathcal{I}_2 = \{1, 1, 1, -1, -1, 1\},$$
$$\mathcal{I}_3 = \{1, -1, -1, -1, -1, 1\}, \; \mathcal{I}_4 = \{-1, -1, -1, -1, -1, -1\}$$

satisfy (3) in Theorem 1.iii. By (13), we obtain the four intervals

$$A_1 = (0.6207, +\infty),\ A_2 = (0.1041, 0.1471),\ A_3 = (0.0005, 0.0012),\ A_4 = (-\infty, -1)$$

and $\hat{A} = -1$ so that $A(p,q) = A_1 \cup A_2 \cup A_3 \cup A_4$. The root loci of $\phi(s,\alpha) = q(s) + \alpha p(s)$ in Figure 1 displays how these four intervals yield Hurwitz stable $\phi(s,\alpha)$.

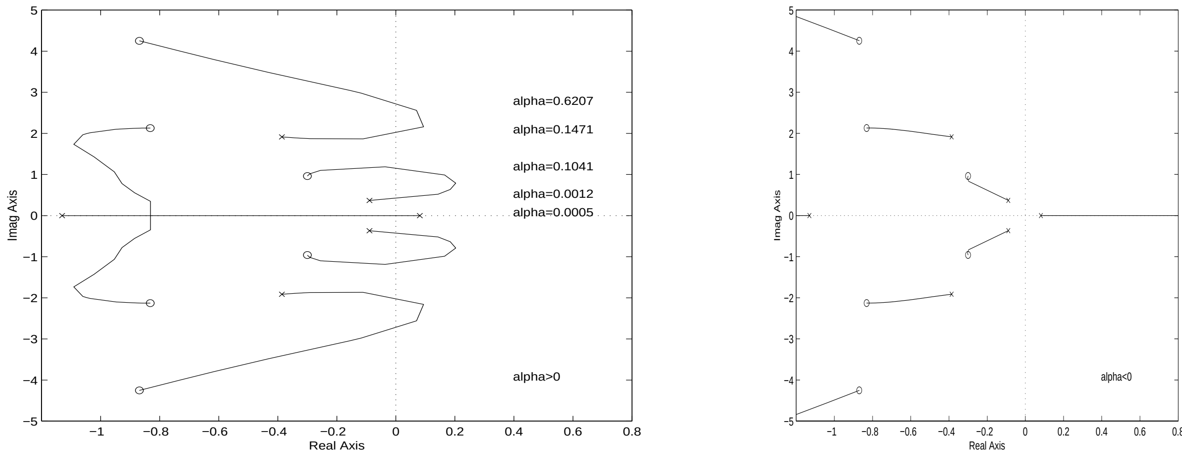

Figure 1: Root-loci of $\phi(s,\alpha)$.

Continuing the same example, we now employ Theorem 2 to determine $B(p,q)$. By the method of Hurwitz determinants, the polynomial q has no zeros in $\mathbf{C}_0$ and $\sigma(q) = 4$. Moreover, $d = 1$ and using (17) we have $D(u) = -G(u)$, $E(u) = H(u)$, $C(u) = u^6 + 6u^5 + 7u^4 - 17.1u^3 - 4.5u^2 - 0.35u + 0.0025$ so that $x_i = v_i$ for $i = 0, ..., 5$. There are 10 different signum sequences $\{i_0, i_1, i_2, i_3, i_4, i_5\}$ that satisfy (2) of Theorem 2.iii, where $n - \sigma(q) = 2$. Now, $D(0_-) = -G(0) = -55 < 0$, $C(x_i) > 0$ for $i = 0, ..., 5$, and

$$\tfrac{E}{C}(x_0) = -2000,\ \tfrac{E}{C}(x_1) = -828.6583,\ \tfrac{E}{C}(x_2) = -9.6063,$$

$$\tfrac{E}{C}(x_3) = -6.7970,\ \tfrac{E}{C}(x_4) = -1.6111,\ \tfrac{E}{C}(x_5) = 1.$$

Three signum sequences

$$\mathcal{I}_1 = \{1, -1, -1, -1, -1, -1\},\ \mathcal{I}_2 = \{1, 1, 1, 1, 1, -1\},$$
$$\mathcal{I}_3 = \{1, 1, 1, -1, -1, -1\}$$

satisfy condition (3) of Theorem 3.iii which yield $B_1 = (828.6583, 2000)$, $B_2 = (-1, 1.6111)$, $B_3 = (6.797, 9.6063)$. The set $\hat{B} = \{-1\}$ and hence $B(p,q) = B_1 \cup B_2 \cup B_3$. The correspondence between $A(p,q)$ and $B(p,q)$ can be seen using Remark 5. •

Example 2. In this example, we illustrate how fixed signums can arise in the candidate signum sequences. Consider

$$q = s^6 + s^5 + 11s^4 + 2s^3 + 19s^2 + 12,$$
$$p = s^5 + 3s^4 + 4s^3 + 6s^2 + 4s.$$

We have $\bar{p} = s^3 + 3s^2 + 2s$, $\sigma(\bar{p}) = 2$, and $G(u) = -(u+1)(u+2)(u+3)(u+4)$, $F(u) = -u(u-1)(u-4)(u+2)$, $H(u) = u(2u^3 + 29u^2 + 53u + 36)$. The zeros of $G(u)$ are $v_0 = 0, v_1 = -1, v_2 = -2, v_3 = -3, v_4 = -4$. Evaluating F at these zeros, $F(v_0) = 0, F(v_2) = 0$. By (1) of Theorem 1.iii, $i_0 = 0$ and $i_2 = 1$. Since $n - \sigma(p) = 4$, the signum sequences $\mathcal{I}_1 = \{0, -1, 1, 1, 1\}$, $\mathcal{I}_2 = \{0, -1, 1, -1, -1\}$, $\mathcal{I}_3 = \{0, 1, 1, -1, 1\}$ are the only ones that satisfy (2) of Theorem 1.iii. Moreover, $\mathcal{S}F(v_3) = \mathcal{S}F(v_4) = -1$ and we have $\mathcal{J}_1 = \{-1, -1, -1\}$, $\mathcal{J}_2 = \{-1, 1, 1\}$, $\mathcal{J}_3 = \{1, 1, -1\}$. Using $G(0_-) < 0$ and

$$\frac{H}{F}(v_1) = -1, \ \frac{H}{F}(v_3) = 3, \ \frac{H}{F}(v_4) = 2,$$

the only signum sequence satisfying the third item of Theorem 1 turns out to be $\mathcal{I}_1$ which yields $A(p,q) = (1, +\infty)$. •

4 Special Cases

In this section, we pursue some consequences of Theorems 1 and 2 and make contact with some results in robust stability analysis. We consider three cases:

(A4) $\quad \bar{p}(s) = 0 \quad \Rightarrow \quad s \in \mathbf{C}_{0+}$.

(A5) $\quad \bar{q}(s) = 0 \quad \Rightarrow \quad s \in \mathbf{C}_{0+}$.

(A6) $\quad q(s) = 0 \quad \Rightarrow \quad s \in \mathbf{C}_{-}$.

By (10) and (16), the polynomials $\bar{p}$ and $\bar{q}$ are free of $\mathbf{C}_0$ roots except possibly a simple root at the origin. Thus, (A4) and (A5) hold if and only if the corresponding polynomial has all its roots in $\mathbf{C}_+$ or one root at 0 and the rest in $\mathbf{C}_+$. Alternatively, (A4) holds if and only if $\sigma(\bar{p}) = -deg\,\bar{p}$ or $\bar{p}(0) = 0$, $\sigma(\bar{p}) = -deg\,\bar{p} + 1$; similarly for (A5). On the other hand (A6) holds if and only $\sigma(q) = n$, or equivalently, $0 \in A(p,q)$, $\beta \in B(p,q)$ as $\beta \to \pm\infty$.

Corollary 1. *Let $p, q \in \mathbf{R}[s]$ satisfy (A1), (A2), (A4). Then, $A(p,q) \neq \emptyset$ if and only if the alternating signum sequence $\mathcal{I} = \{G(0)(-1)^j\}$ satisfies (1)-(3) in Theorem 1, in which case*

$$A_1 = \begin{cases} \left(-\min\limits_{j \in \mathcal{J}^+} \frac{H}{F}(v_j), \ -\max\limits_{j \in \mathcal{J}^-} \frac{H}{F}(v_j)\right) & \text{if} \quad G(0) > 0 \\ \\ \left(-\min\limits_{j \in \mathcal{J}^-} \frac{H}{F}(v_j), \ -\max\limits_{j \in \mathcal{J}^+} \frac{H}{F}(v_j)\right) & \text{if} \quad G(0) < 0, \end{cases} \tag{23}$$

where $\mathcal{J}^+ := \{j : \ \mathcal{S}F(v_j) = (-1)^j\}$ and $\mathcal{J}^- := \{j : \ \mathcal{S}F(v_j) = (-1)^{j+1}\}$,

$$\hat{A} = \begin{cases} \emptyset & \text{if} \quad n > m \\ \{-\frac{p}{q}(\infty)\} & \text{if} \quad n = m, \end{cases}$$

and $A(p,q) = A_1 \setminus (\hat{A} \cap A_1)$.

Proof. Let $\bar{p}$ satisfy (A4). Let us first consider the case $\sigma(p) = \sigma(\bar{p}) = \Leftrightarrow deg\,\bar{p}$. If $n \Leftrightarrow m$ is odd, then by (7), we have $2deg\,G = n + deg\,\bar{p} \Leftrightarrow 1$ so that $n \Leftrightarrow \sigma(p) = 2deg\,G + 1$. As $2k+1$ is the maximum value that can be attained in the right hand side of (2) of Theorem 1, (2) can be fulfilled if and only if $k = deg\,G$. Hence condition (iii) in Theorem 1 can be satisfied if and only if $k = deg\,G$ (i.e., all roots of G are real negative and distinct) and $\mathcal{I}$ is the alternating sequence with elements $i_j = G(0)(\Leftrightarrow 1)^j$ for $j = 0, 1, ...$ If $n \Leftrightarrow m$ is even, then by (7), $2deg\,G + 1 \leq n + deg\,\bar{p} \Leftrightarrow 1$ so that $n + deg\,\bar{p} \Leftrightarrow 1 \geq 2deg\,G + 1 \geq 2k+1$. As $2k+2$ is the maximum number that can be attained in the right hand side of (2) of Theorem 1, (2) can be fulfilled if and only if $k = deg\,G$ and $\mathcal{I} = \{G(0)(\Leftrightarrow 1)^j\}$. In the case $\bar{p}(0) = 0$ and $\sigma(p) = \sigma(\bar{p}) = \Leftrightarrow deg\,\bar{p} + 1$ we have $i_0 = 0$ and by similar arguments we again have that (2) holds if and only if $k = deg\,G$ and $i_j = G(0)(\Leftrightarrow 1)^j$ for $j = 1, 2, ...$ Since G has all its roots distinct, the set $\{u_j\}$ is empty and $\hat{A}$ can have at most one element. Hence, for the alternating signum sequences the set $A(p,q)$ simplifies to $A(p,q) = A_1 \setminus (\hat{A} \cap A_1)$, where A_1 is given by (23). □

Corollary 2. *Let* $p, q \in \mathbf{R}[s]$ *satisfy (A1), (A2), (A5). Then,* $B(p,q) \neq \emptyset$ *if and only if the alternating signum sequence* $\mathcal{I} = \{D(0)(\Leftrightarrow 1)^j\}$ *satisfies (1)-(3) in Theorem 2, in which case*

$$B_1 = \begin{cases} (\Leftrightarrow \min\limits_{j \in \mathcal{J}^+} \dfrac{E}{C}(x_j),\ \Leftrightarrow \max\limits_{j \in \mathcal{J}^-} \dfrac{E}{C}(x_j)) & \text{if } D(0) > 0 \\ (\Leftrightarrow \min\limits_{j \in \mathcal{J}^-} \dfrac{E}{C}(x_j),\ \Leftrightarrow \max\limits_{j \in \mathcal{J}^+} \dfrac{E}{C}(x_j)) & \text{if } D(0) < 0, \end{cases}$$

where $\mathcal{J}^+ := \{j : \ \mathcal{SC}(v_j) = (\Leftrightarrow 1)^j\}$ *and* $\mathcal{J}^- := \{j : \ \mathcal{SC}(v_j) = (\Leftrightarrow 1)^{j+1}\}$,

$$\hat{B} = \begin{cases} \emptyset & \text{if} \quad n > m \\ \{\Leftrightarrow \frac{q}{p}(\infty)\} & \text{if} \quad n = m, \end{cases}$$

and $B(p,q) = B_1 \setminus (\hat{B} \cap B_1)$.

Proof. The proof is similar to the proof of Corollary 1 and it is omitted. □

If $n > m$, then $\hat{A} = \emptyset$ and $\hat{B} = \emptyset$ in Corollaries 1 and 2. It follows that if either (A4) or (A5) holds, then the set $A(p,q)$ is an interval (possibly empty). Consequently, the pair of polynomials (p,q) has the following property:

(CC) $q + \alpha_1 p,\ q + \alpha_2 p \in \mathcal{H}$ for some $\alpha_1 < \alpha_2$ in $\mathbf{R} \ \Rightarrow \ q + \alpha p \in \mathcal{H} \ \ \forall \ \ \alpha \in [\alpha_1, \alpha_2]$.

The condition (CC) is a *convexity condition* for $(q + \mathbf{R}p) \cap \mathcal{H}$, where $(q + \mathbf{R}p) := \{q + \alpha p : \alpha \in \mathbf{R}\}$. We refer the reader to [11], [5] for motivations of studying (CC) when q is a stable polynomial. We note that (CC) is a slight generalization (to unstable q) of the *geometric local concept of convex directions* introduced in [5]. Of particular relevance to (CC) is Theorem 2 of [11], which gives a necessary and sufficient condition on p in order for (p,q) to satisfy (CC) *for any* Hurwitz stable q.

By Corollaries 1 and 2, *if either p satisfies (A4) or q satisfies (A5), then a coprime pair (p,q) with $\deg q > \deg p$ satisfies (CC).* Note that if p (resp. q) is an even or odd polynomial in s or a polynomial having all its roots in $\mathbf{C}_+$, or if it is a multiple of polynomials of these two types, then p (resp. q) satisfies (A4) (resp. (A5)) and the pair (p,q) satisfies (CC) *for any* q (resp. p) such that $n > m$.

These simple conditions obtained by Corollaries 1 and 2 are only sufficient conditions for (CC) to hold. Theorem 1 (or Theorem 2) of course yields a necessary and sufficient condition on posing the requirement that at most one signum sequence satisfying the conditions (1) and (2) of Theorem 1 also satisfies condition (3). In order to cut down the number of different signum sequences which must be tested however, we further investigate this question below under the simplifying assumption (A6), i.e., we assume that q is Hurwitz stable. Under this assumption, the condition obtained by Theorem 2 can be considerably simplified and Corollary 4 below yields a necessary and sufficient condition for $A(p,q)$ to consist of exactly one interval. The condition obtained is very easy to check.

Let (A6) hold. Then, $q \in \mathcal{H}$ so that $\sigma(q) = n$. As q is free of $\mathbf{C}_0$ zeros, we have $\bar{q} = q$ and $q(0) \neq 0$. Since $\beta \in B(p,q)$ for $\beta \to \pm\infty$, the conditions (i)-(ii) of Theorem 2 hold and since $C(u) > 0$ for all $u \leq 0$, the condition (1) of (iii) is trivially satisfied. Thus, all elements of the candidate signum sequences $\mathcal{I}$ of Theorem 2 are free and they should satisfy

$$0 = i_0 + (-1)^{k+1} i_{k+1} + 2[(i_2 + i_4 + ...) - (i_1 + i_3 + ...)]. \tag{24}$$

In particular, the constant signum sequences $\{+1\}$ and $\{-1\}$ satisfy (24) yielding the intervals $(-\infty, -b_2)$ and $(-b_1, \infty)$ with

$$b_1 := \min_j \frac{E}{C}(x_j), \;\; b_2 := \max_j \frac{E}{C}(x_j). \tag{25}$$

There may of course be other signum sequences satisfying (iii) of Theorem 1. Below in Corollary 3, we simplify the condition of Theorem 2 for the existence of such sequences and associated intervals. Let

$$\beta_j := \frac{E}{C}(x_j), \;\; j = 0, 1, ..., k+1. \tag{26}$$

We order β_j as

$$\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_{k+2}} \tag{27}$$

where β_{j_λ} occurs to the left of β_{j_κ} (i.e., $\lambda < \kappa$) if and only if either $\beta_{j_\lambda} < \beta_{j_\kappa}$ or $\beta_{j_\lambda} = \beta_{j_\kappa}$ and $j_\lambda < j_\kappa$. Note that $\beta_{j_1} = b_1$ and $\beta_{j_{k+2}} = b_2$, by (25). Let us denote

$$M(t) := \max\{l : \beta_{j_l} = \beta_{j_t}\}. \tag{28}$$

for $t = 1, ..., k+2$ and let μ, ν be such that

$$j_\mu = 0, \;\; j_\nu = k+1. \tag{29}$$

Example 3. If $k = 7$ and

$$\begin{aligned} &\beta_0 = 0.5,\ \beta_1 = 0.2,\ \beta_2 = 1,\ \beta_3 = 4.1,\ \beta_4 = 1, \\ &\beta_5 = 0.2,\ \beta_6 = -3,\ \beta_7 = 1,\ \beta_8 = -3, \end{aligned}$$

then

$$\begin{aligned} &j_1 = 6,\ j_2 = 8,\ j_3 = 1,\ j_4 = 5,\ j_5 = 0, \\ &j_6 = 2,\ j_7 = 4,\ j_8 = 7,\ j_9 = 3 \\ &M(1) = M(2) = 2,\ M(3) = M(4) = 4, \\ &M(6) = M(7) = M(8) = 8,\ M(5) = 5,\ M(9) = 9, \\ &\mu = 5,\ \nu = 2. \end{aligned}$$

We also note that for this case the condition (30) below fails. •

Corollary 3. *Let $p, q \in \mathbf{R}[s]$ satisfy (A1), (A2), (A6) and let β_j, j_t, $M(t)$, and μ, ν be as in (26)-(29). For some $t = 1, ..., k+1$, the interval*

$$B_t := (-\beta_{j_{t+1}}, -\beta_{j_t})$$

is such that $B_t \setminus (\hat{B} \cap B_t)$ is contained in $B(p,q)$ if and only if $t = M(t)$, max $\{\mu, \nu\} \leq t$ or min $\{\mu, \nu\} > t$, and

$$\begin{cases} \displaystyle\sum_{l=1}^{M(t)} (j_l \bmod 2) = \frac{M(t)}{2} & \text{if } \min\{\mu, \nu\} > t \\ \displaystyle\sum_{l=M(t)+1}^{k+2} (j_l \bmod 2) = \frac{k - M(t)}{2} + 1 & \text{if } \max\{\mu, \nu\} \leq t. \end{cases} \tag{30}$$

Proof. Note that the distinct values in $\{\beta_j\}$ are $\{\beta_{j_{M(t)}}, \beta_{j_{M(t)+1}}\}$. By (A6), the constant signum sequences yield the intervals $(-\infty, -\beta_{j_{M(k+2)}})$ and $(-\beta_{j_{M(1)}}, \infty)$ which are (except their common points with $\hat{B}$) contained in $B(p,q)$.

[**Only if**] We assume that $D(0_-) > 0$ as the case $D(0_-) < 0$ is similar. Suppose for some $t = 1, ..., k+1$, $B_t \setminus (\hat{B} \cap B_t)$ is contained in $B(p,q)$. By Theorem 2, there exists $\mathcal{I}_t = \{i_j\}$ satisfying (24) and

$$\beta_{j_t} = \max_{\{j:\, i_j = -1\}} \beta_j,\ \ \beta_{j_{t+1}} = \min_{\{j:\, i_j = 1\}} \beta_j. \tag{31}$$

By Remark 4, $\mathcal{I}_t$ can not be a constant sequence so that β_{j_t} and $\beta_{j_{t+1}}$ are both finite values and $t \in \{1, ..., k+1\}$. Moreover, as $\beta_{j_t} \neq \beta_{j_{t+1}}$, it must be that $t = M(t)$. By (31) and by the definition of the index i_j,

$$i_{j_l} = \begin{cases} 1 & \text{for } l > t \\ -1 & \text{for } l < t+1. \end{cases} \tag{32}$$

In order for (24) to be satisfied, i_0 and i_{k+1} should have the same sign (whether k is even or odd). Hence, 0, $k+1 \in \{i_{j_1}, ..., i_{j_t}\}$ or 0, $k+1 \in \{i_{j_{t+1}}, ..., i_{j_{k+2}}\}$. Equivalently, $max\,\{\mu, \nu\} \leq t$ or $min\,\{\mu, \nu\} > t$. If we let $n_{e(o)}$ denote the number of even (odd) integers in $\{j_1, ..., j_t\}$ and let $m_{e(o)}$ denote the number of even (odd) integers in $\{j_{t+1}, ..., j_{k+2}\}$, then (24) and (32) yield

$$\begin{array}{lll} 0 = \Leftrightarrow n_e + m_e + n_o \Leftrightarrow m_o & \text{if} \;\; k \;\text{ is even} & \\ 0 = \Leftrightarrow n_e + m_e + n_o \Leftrightarrow m_o \Leftrightarrow 1 & \text{if} \;\; k \;\text{ is odd and} & max\,\{\mu, \nu\} \leq t \\ 0 = \Leftrightarrow n_e + m_e + n_o \Leftrightarrow m_o + 1 & \text{if} \;\; k \;\text{ is odd and} & min\,\{\mu, \nu\} > t. \end{array}$$

We now note that $n_e + n_o = t$, $m_e + m_o = k + 2 \Leftrightarrow t$, $n_e + m_e = (k+2)/2$ if k is even, and $n_e + m_e = (k+3)/2$ if k is odd. Using these above, we obtain $n_o = t/2$ if k is even, $m_o = (k \Leftrightarrow t+2)/2$ if k is odd and $max\,\{\mu, \nu\} \leq t$, and $n_o = t/2$ if k is odd and $min\,\{\mu, \nu\} > t$. Hence, one of (30) holds.

[If] If $t = M(t)$ exists such that (30) holds, then let

$$i_{j_l} := \begin{cases} \Leftrightarrow D(0_-) & \text{for } \; l \leq M(t) \\ D(0_-) & \text{for } \; l > M(t) \end{cases}$$

It is straightforward to check that $\mathcal{I}_t = \{i_j\}$ satisfies (24) and yields the interval $B_{M(t)}$. □

Remark 6. An equivalent way of stating (30) using the notation introduced in the above proof is "$min\{\mu, \nu\} > t \;\; \Rightarrow \;\; n_o = n_e > 0$ *and* $max\{\mu, \nu\} \leq t \;\; \Rightarrow \;\; m_o = m_e > 0$." △

Corollary 4. *Let (A1), (A2), (A6) hold. Then, $A(p, q) = (\Leftrightarrow \alpha_1, \alpha_2)$ for some positive numbers α_1, α_2 [or equivalently, $B(p, q) = (\Leftrightarrow \infty, \Leftrightarrow \alpha_1^{-1}) \cup (\alpha_2^{-1}, \infty)$] if and only if $\hat{B} \subset [\beta_{j_1}, \beta_{j_{k+2}}]$ and for $t = 1, ..., k+1$ it holds that*

$$\begin{array}{ll} \displaystyle\sum_{l=1}^{M(t)} (j_l \; mod \; 2) \neq \frac{M(t)}{2} & \textit{for all } M(t) = t \textit{ such that } \; min\,\{\mu, \nu\} > t \\ \textit{and} & \\ \displaystyle\sum_{l=M(t)+1}^{k+2} (j_l \; mod \; 2) \neq \frac{k \Leftrightarrow M(t)}{2} + 1 & \textit{for all } M(t) = t \textit{ such that } \; max\,\{\mu, \nu\} \leq t. \end{array} \qquad (33)$$

Proof. This is an immediate consequence of Corollary 3. □

Example 4. We consider Example 4.4 in [5]. Let $q(s) = (s+1)^3$ and $p(s) = s^2 + p_1 s + p_0$, where $p_0, p_1 \in \mathbf{R}$. We use the result of Corollary 4 to determine the set of values (p_1, p_0) for which (p, q) satisfies the condition (CC). By an easy computation $D(u) = \Leftrightarrow u^2 + (3p_1 \Leftrightarrow p_0 \Leftrightarrow 3)u + (p_1 \Leftrightarrow 3p_0)$ and $D(u)$ has two negative real distinct zeros if and only if (p_1, p_0) are such that

$$3 + p_0 \Leftrightarrow 3p_1 > 0, \qquad (34)$$
$$3p_0 \Leftrightarrow p_1 > 0, \qquad (35)$$
$$\Delta := p_0^2 \Leftrightarrow 6(1 + p_1)p_0 + 9 + 9p_1^2 \Leftrightarrow 14p_1 \geq 0 \qquad (36)$$

Case 1: If one or more of (34)-(36) fail, then $k \leq 1$ and the condition of Corollary 4 is easily seen to be satisfied. **Case 2:** If (34)-(36) all hold with $\Delta > 0$ then D has two real negative and distinct roots $x_1 = 0.5(3p_1 - p_0 - 3 + \sqrt{\Delta}) > x_2 = 0.5(3p_1 - p_0 - 3 - \sqrt{\Delta})$. In this case,

$$\beta_0 = p_0,\ \beta_1 = \begin{cases} \frac{x_1+p_0}{3x_1+1}, & x_1 \neq -\frac{1}{3} \\ \frac{p_1}{x_1+3}, & x_1 \neq -3 \end{cases},\ \beta_2 = \begin{cases} \frac{x_2+p_0}{3x_2+1}, & x_1 \neq -\frac{1}{3} \\ \frac{p_1}{x_2+3}, & x_2 \neq -3 \end{cases},\ \beta_3 = 0. \tag{37}$$

The statement of Corollary 4 simplifies to *A(p,q) is not an interval if and only if*

$$max\{\beta_1, \beta_2\} < min\{0, p_0\} \quad or \quad min\{\beta_1, \beta_2\} > max\{0, p_0\}. \tag{38}$$

By an easy computation $\beta_1\beta_2 = p_1/8$. If $p_1 \leq 0$, then $\beta_1\beta_2 \leq 0$ and (38) fails. If $p_1 > 0$, then (36) can be written as $\Delta = [p_0 - 3(1 + p_1) + 4\sqrt{2p_1}][p_0 - 3(1 + p_1) - 4\sqrt{2p_1}] > 0$ and we only need to consider two cases : **Case 2.1:** $p_1 > 0$, $p_0 > 3(1 + p_1) + 4\sqrt{2p_1}$. In this case (34) and (35) are trivially satisfied. Moreover $x_1 + 3 < 0$ which implies that $max\{\beta_1, \beta_2\} < 0$. Hence, (38) holds and $A(p, q)$ is not an interval. (It can be seen that $A(p, q) = (-1/p_0, -1/\beta_1) \cup (-1/\beta_2, +\infty)$ using Corollary 3 and Remark 5. Note that the additional interval is contained in the positive real axis.) The set of (p_1, p_0) satisfying $p_1 > 0,\ p_0 > 3(1 + p_1) + 4\sqrt{2p_1}$ is the shaded region of the first figure in Figure 2. **Case 2.2:** If $p_1 > 0$ and $p_0 < 3(1 + p_1) - 4\sqrt{2p_1}$, $\beta_1 + \beta_2 = (3p_1 - p_0 + 3)/8 > 0$. By (35), $p_0 > 0$ and hence (38) implies $max\{\beta_1, \beta_2\} > p_0$. Hence, by (37), we have $(3x_1 + 1)(3x_2 + 1) > 0$ which implies $p_0 > 1/3$. Using $\beta_1\beta_2 = p_1/8 > p_0^2$, $\beta_1 + \beta_2 > 2p_0$ and (35) it follows that $8/9 < p_1 < 9/8$ and $1/3 < p_0 < 3/8$. In p_1p_0-plane, the region determined by $8/9 < p_1 < 9/8$, $1/3 < p_0$, $3 + p_0 - 3p_1 > 0$ is a small region in the lower right hand side of the first figure in Figure 2 which is magnified in the second figure in Figure 2. For these parameter values $A(p, q)$ consists of two intervals : $A(p, q) = (-1/\beta_1, +\infty) \cup (-1/\beta_2, -1/p_0)$. The additional interval is contained in the negative real axis. **Case 3:** If (34)-(36) all hold with $\Delta = 0$ then D has a real negative root with multiplicity two ($x_1 = x_2$). Then $A(p, q)$ is an interval if and only if $\hat{A} = \{-1/\beta_1\}$ is not included in $(-1/p_0, +\infty)$. We note that $\Delta = 0$ only when $p_1 \geq 0$. Suppose that $p_1 = 0$. Then $p_0 = 3$ and $\beta_1 = 0$. Let $p_1 > 0$. We need to consider two cases : **Case 3.1:** $p_1 > 0$, $p_0 = 3(1 + p_1) + 4\sqrt{2p_1}$. Then $-1/\beta_1 > 0$, hence $\hat{A}$ is in $(-1/p_0, +\infty)$. **Case 3.2:** $p_1 > 0$, $p_0 = 3(1 + p_1) - 4\sqrt{2p_1}$. We obtain that $-1/\beta_1 > -1/p_0$ when $8/9 < p_1 < 9/8$ by using similar arguments as in Case 2.1. Consequently, (p, q) *satisfies (CC) if and only if* (p_1, p_0) *is not in* $\{(p_1, p_0) : p_1 > 0,\ p_0 \geq 3(1 + p_1) + 4\sqrt{2p_1}\} \cup \{(p_1, p_0) : p_1 > 0,\ 1/3 < p_0 \leq 3(1 + p_1) - 4\sqrt{2p_1},\ 8/9 < p_1 < 9/8\}$. •

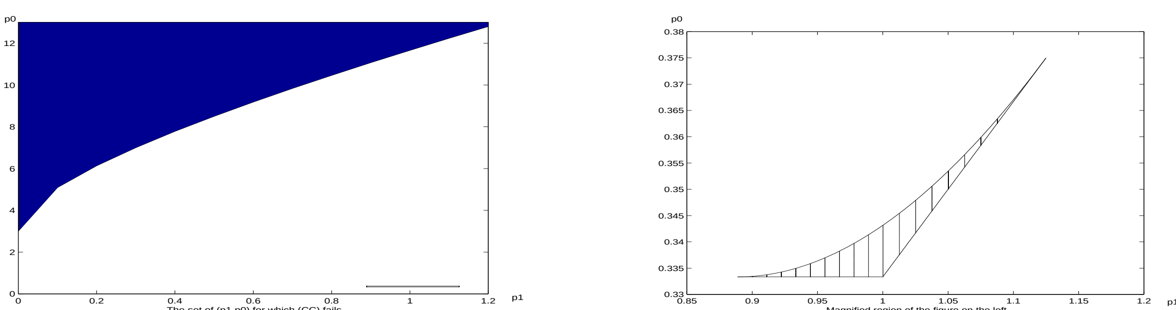

Figure 2: For the shaded regions (CC) fails.

Corollary 5. *Suppose (A1), (A2), (A6) hold and let $n > m$. Let*

$$\tilde{D} = D/gcd\{D,E\},\ \tilde{E} = E/gcd\{D,E\},$$

and suppose all the real negative zeros of $\tilde{D}$ have odd multiplicities. Let $\{\tilde{x}_1,...,\tilde{x}_{\tilde{k}}\}$ be the real negative zeros of $\tilde{D}$. If the sequence $\{\mathcal{S}\tilde{E}(\tilde{x}_1),...,\mathcal{S}\tilde{E}(\tilde{x}_{\tilde{k}})\}$ is alternating, then $A(p,q)$ is an interval.

Proof. By (A6), we have $b = 1$ and $C(x_j) > 0$ for $j = 0,1,...,k+1$. A real negative zero u of D is either a zero of $\tilde{D}$ or of $\hat{D} := gcd\{D,E\}$. Let the real negative zeros with odd multiplicities of $\hat{D}$ be $\hat{x}_1,...,\hat{x}_{\hat{k}}$. We have

$$\mathcal{S}\beta_j = \mathcal{S}\frac{E}{C}(x_j) = \mathcal{S}\tilde{E}(x_j)\mathcal{S}\hat{D}(x_j),\ j = 1,...,k.$$

Since all the negative zeros of even multiplicity of D are also zeros of $\hat{D}$, and since $n > m$, the set $\hat{B}$ is equal to $\{0\}$. Consider $\tilde{x}_i$ for which $\hat{D}(\tilde{x}_i) \neq 0$ for $i = 1,\ldots,\tilde{k}$. Let n_i denote the number of those elements in $\{\hat{x}_j\}\setminus(\{\tilde{x}_j\}\cap\{\hat{x}_j\})$ that are greater than $\tilde{x}_i$. Let m_i denote the number of those elements in $\{\tilde{x}_j\}\cap\{\hat{x}_j\}$ that are greater than $\tilde{x}_i$. It is easy to see that, if $x_j = \tilde{x}_i$, then $j + m_i = i + n_i$ and hence $\mathcal{S}\hat{D}(x_j) = (-1)^{n_i+m_i}\mathcal{S}\hat{D}(0_-)$. On the other hand, $\mathcal{S}\tilde{E}(x_j) = \mathcal{S}\tilde{E}(\tilde{x}_i) = (-1)^{(i-1)}\mathcal{S}\tilde{E}(\tilde{x}_1)$ as the sequence $\{\tilde{E}(\tilde{x}_i)\}$ is alternating. Consequently, if $\hat{D}(x_j) \neq 0$, then $\mathcal{S}\beta_j = (-1)^{j-1+2m_i}\mathcal{S}\tilde{E}(\tilde{x}_1)\mathcal{S}\hat{D}(0_-)$ for $j = 1,...,k$; or assuming $\mathcal{S}\tilde{E}(\tilde{x}_1)\mathcal{S}\hat{D}(0_-) = 1$ without loss of generality,

$$\mathcal{S}\beta_j = \begin{cases} 1 & \text{if } j \text{ is odd and } \hat{D}(x_j) \neq 0 \\ 0 & \text{if } \hat{D}(x_j) = 0 \\ -1 & \text{if } j \text{ is even and } \hat{D}(x_j) \neq 0 \end{cases}$$

for $j = 1,...,k$. Since $n > m$, $\beta_{k+1} = 0$. Using the notation introduced in the proof of Corollary 3, for $t < min\{\mu,\nu\}$ we have $n_o = 0$, and for $t \geq max\{\mu,\nu\}$ we have $m_e = 0$ which implies by Remark 6 that $A(p,q)$ is an interval.

As an example of (p,q) satisfying the condition of Corollary 5, we can mention any pair (p,q), where $\bar{p}$ has all its roots in $\mathbf{C}_{0+}$[5]. As another example, consider (p,q) for any Hurwitz stable q and $\bar{p}$ with $deg\,\bar{p} = 1 < deg\,q$. Using Lemma 1 and the fact that $n - \sigma(p) = n - 1$, it is easy to see that for such $\bar{p}$, all zeros of G are real, negative, and distinct and the sequence $\{H(v_1),...,H(v_k)\}$ is alternating. Moreover, $gcd\{D,E\} = \hat{D} = d$ and $\tilde{D} = -G$, $\tilde{E} = H$ so that Corollary 5 yields: *If $\bar{p}$ is a Hurwitz stable polynomial of degree one, then for any Hurwitz stable q with higher degree than $deg\,p$ condition (CC) holds.* We note that, these examples for the classes of polynomials satisfying (CC) *for any Hurwitz stable q* can also be obtained via Theorem 2 of [11].

[5] Note that for this case Corollary 1 yields a stronger result.

5 Conclusions

In Theorems 1 and 2, we have obtained an analytic method for the existence and determination of stabilizing feedback gains. The methods can be viewed as analytic versions of the Nyquist and the inverse Nyquist methods and they are dual to each other in the same way as the Nyquist and the inverse Nyquist methods are. The link between the two methods is established in Remark 5. The discrete-time version of Hurwitz stability, the Schur stability, can be developed in a similar manner but the details have to be worked out.

Computationally, the methods of Theorem 1 or 2 can be compared with the Neimark D-decomposition method. In the latter, one is required to apply some algebraic stability test (such as the Routh array method) in each predetermined interval on the real axis. In the former, this burden is replaced by the determination of all signum sequences satisfying (1) and (2) in the theorem statements. The number of such sequences can be quite large. One remedy for this is to exploit the connection in Remark 5 to cut down the number of candidate signum sequences still further. In fact if $\{i_j\}$ satisfies (1) and (2) in Thorem 1, then the transformed signum sequence $\{\bar{i}_j\}$ of (22) should satisfy (1) and (2) of Theorem 2, which puts a further constraint on $\{i_j\}$ reducing the number of signum sequences which must be tested for condition (3). The details of this reduction is left for future work.

Theoretically, the obtained methods yield results in the relatively new areas of research. The main results of Section 3 are all new results in the study of convex directions. Under the simplifying assumption that q is Hurwitz stable, we have obtained a very simple test for a pair (p, q) to satisfy the convexity condition (CC). Whether a complete characterization of all pairs (p, q) satisfying (CC) can be obtained from Theorems 1 and 2 is another open question.

6 Appendix: Proof of Lemma 1.

We first consider the case $\psi(0) \neq 0$. Since (a, b) is coprime, in this case ψ has no zeros on $\mathbf{C}_0$ and $a(0) \neq 0$. Let the real negative roots (if any) with odd multiplicities of $a(u)$ be [6]

$$u_1 > u_2 > \cdots > u_l$$

and define

$$U := \begin{cases} \{u_j\}_{j=1}^{l} & \text{if } m \text{ is even} \\ \{u_j\}_{j=1}^{l} \bigcup \{u_{l+1} = \Leftrightarrow\infty\} & \text{if } m \text{ is odd,} \end{cases} \tag{39}$$

$$V := \begin{cases} \{v_i\}_{i=1}^{k} \bigcup \{v_0 = 0,\, v_{k+1} = \Leftrightarrow\infty\} & \text{if } m \text{ is even} \\ \{v_i\}_{i=1}^{k} \bigcup \{v_0 = 0\} & \text{if } m \text{ is odd,} \end{cases} \tag{40}$$

where $m := deg\,\psi$. We now order the elements of $U \cup V$ as

$$0 = t_1 > t_2 > \cdots > t_{k+l+2} = \Leftrightarrow\infty$$

[6]The notation in the appendix deviates from that of the main text.

and define the index sets I and J which distinguishes certain elements in $\{t_j\}$:

$$\begin{array}{lll} i \in I \Leftrightarrow \;\; t_i \in V \text{ and } t_{i+1} \in U & \text{for} & i = 1, 2, \ldots, k+l+1, \\ j \in J \Leftrightarrow t_j \in U \text{ and } t_{j+1} \in V & \text{for} & j = 1, 2, \ldots, k+l+1. \end{array}$$

By either tracing the Leonhard locus[7] of $\psi(j\omega)$ ([7], §V.1) or by Cauchy index ([3], XV.3) considerations, it is now easy to compute the net change in $\phi(\omega) := arg\, \psi(j\omega)$ as ω increases from 0 to ∞ as

$$\Delta_0^\infty \; \phi(\omega) = \frac{\pi}{2}(\sum_{i \in I} \mathcal{S}a(t_i)\mathcal{S}b(t_{i+1}) \Leftrightarrow \sum_{j \in J} \mathcal{S}b(t_j)\mathcal{S}a(t_{j+1})).$$

By ([3], §XV.3), $\sigma(\psi) = \frac{2}{\pi}\Delta_0^\infty \; \phi(\omega)$ and we obtain

$$\sigma(\psi) = \sum_{i \in I} \mathcal{S}a(t_i)\mathcal{S}b(t_{i+1}) \Leftrightarrow \sum_{j \in J} \mathcal{S}b(t_j)\mathcal{S}a(t_{j+1}). \tag{41}$$

We now show that the right hand sides of (9) and (41) are the same. Suppose first that $deg(\psi)$ is even. The right hand side of (9) can be written as

$$\mathcal{S}b(0_-)\sum_{i=0}^{k}((\Leftrightarrow 1)^i(\mathcal{S}a(v_i) \Leftrightarrow \mathcal{S}a(v_{i+1})). \tag{42}$$

Let μ_i denote the number of $\{u_j\}$ between v_i and v_{i+1} for $i = 0, 1, \ldots, k+1$. Hence, we can rewrite (42) as

$$\mathcal{S}b(0_-)\sum_{i=0}^{k} 2(\mu_i \; mod \; 2)(\Leftrightarrow 1)^i \mathcal{S}a(v_i). \tag{43}$$

On the other hand, the right hand side of (41) can be written as

$$\sum_{i: u_i \neq 0} (\mathcal{S}a(v_i)\mathcal{S}b(v_{i-}) \Leftrightarrow \mathcal{S}b(v_{i-})\mathcal{S}a(v_{i+1}) \;). \tag{44}$$

By noting that $\mathcal{S}a(v_i) = \mathcal{S}a(v_{i+1})$ when μ_i is even for $i = 0, 1, \ldots, k$, we obtain that

$$\sigma(\psi) = \sum_{i \,:\, u_i \; odd} 2\mathcal{S}a(v_i)\mathcal{S}b(v_{i-}). \tag{45}$$

We also have $\mathcal{S}b(v_{i-}) = (\Leftrightarrow 1)^i \mathcal{S}b(0_-)$, since $b(\cdot)$ have i zeros between v_{i-} and 0_- for $i = 0, 1, \ldots, k$. Hence, the right hand sides of (43) and (45) are equal. For the case $deg(\psi)$ is odd, the equality of the right hand sides of (9) and (41) can be shown similarly.

We now consider the case $\psi(0) = 0$. In this case by coprimeness of (a, b), $\psi(s)$ has a simple zero at the origin. Using

$$\sigma(\psi) = \frac{2}{\pi}\Delta_{0+}^\infty \; \phi(\omega)$$

[7] In the Russian literature, this is known as the Michailov plot.

and repeating all the above arguments by appropriate modifications it is possible to show that r given by (9) is again equal to $\sigma(\psi)$. Here we only give a heuristic argument. Let $\tilde{a}$ be a polynomial obtained by a slight perturbation of the coefficients of a and let $\tilde{\psi} := \tilde{a}(s^2) + sb(s^2)$. If the perturbations are sufficiently small, then $\tilde{\psi}$ is such that $\mathcal{S}a(v_i) = \mathcal{S}\tilde{a}(v_i)$ for $i = 1, ..., k+1$ and the root at $s = 0$ of ψ moves either to $\mathbf{C}_-$ or to $\mathbf{C}_+$. In either case, $\tilde{r} := \sigma(\tilde{\psi}) = r \pm 1$. By what has been proved, (9) holds with r, a replaced by $\tilde{r}, \tilde{a}$. Using the fact that $\mathcal{S}a(v_i) = \mathcal{S}\tilde{a}(v_i)$ for $i = 1, ..., k+1$, we obtain that (9) holds with $\mathcal{S}a(0) = 0$. □

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