A SYSTEMIC APPROACH TO PARAUNITARY FILTER BANK DESIGN

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ABSTRACT

This paper deals with the problem of designing paraunitary filter banks adapted to signal statistics. Firstly, an optimum compaction filter is designed for the first channel of the filter bank. Then, the remaining filters are obtained by computing an orthogonal completion for the state space realization of the polyphase transfer function associated with the compaction filter. This approach for the filter bank completion is efficient and has excellent numerical properties.

1. INTRODUCTION

The design of paraunitary FIR filter banks adapted to signal statistics has received significant attention in the latest years. One interesting design method has been proposed by Moulin and Mihçak [1], with the purpose of maximizing the coding gain of the filter bank. In the first and the most difficult step of this method, an optimum compaction filter $H_1(z)$ is obtained for the first channel of the filter bank (the analysis bank is represented in Fig. 1). An optimum compaction filter maximizes the energy of the output signal y_1, for a wide sense stationary input x, with specified second order statistics. The filters for the remaining $m-1$ channels are determined using the eigenvectors of a (transformed) correlation matrix. (The synthesis filters are simply the reversed of the analysis filters.)

In this paper we propose a new approach to the problem. The optimum compaction filter is determined using a low complexity semidefinite programming formulation; the details of the algorithm are given in [2]. The main contribution of this paper is the computation of the remaining filters, based on a state-space representation of the polyphase matrix associated with the filter bank. The resulting algorithm has low complexity and good numerical properties.

It was remarked by Kirac and Vaidyanathan [3] that the algorithm from [1] does not maximize the coding gain. We present new considerations about the optimality and also experimental results.

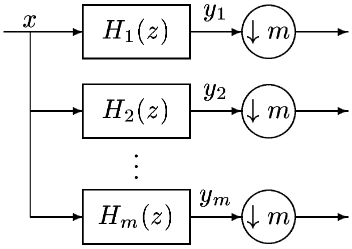

Figure 1: The analysis part of a m-channel filter bank.

Let us introduce now the main notations. Let $x(t)$ be a discrete time wide sense stationary stochastic signal with autocorrelation sequence defined by $r(k) = E[x(t)x(t-k)]$, with $r(0) = 1$. The associated autocorrelation matrix $R = \text{Toep}(r(0), r(1), \ldots, r(N))$, truncated at size $N+1$, is Toeplitz symmetric. Let m be the number of channels of the FIR filter bank and suppose that $N+1$ is taken as a multiple of m, i.e. $N+1 = m(n+1)$.

The filters $H_i(z)$, $i = 1:m$, are FIR of order N, with impulse response $h_i(k)$, $k = 0:N$, that is $H_i(z) = \sum_{k=0}^{N} h_i(k) z^{-k}$; we also see the sequence h_i as a unit norm vector in the Euclidian space $\mathcal{R}^{N+1}$. Putting these vectors as the columns of an $(N+1)\times m$ matrix, we will denote

$$H = [h_1 \; h_2 \; \ldots \; h_m]. \tag{1}$$

The energy of the signals y_i at the output of filters $H_i(z)$ is $\sigma_i^2 = h_i^T R h_i$. The coding gain of the filter bank is (the energy of the input signal is 1 and the filter bank is paraunitary, i.e. conserves the total energy)

$$G_c(H) = \frac{\frac{1}{m}\sum_{i=1}^{m}\sigma_i^2}{\left(\prod_{i=1}^{m}\sigma_i^2\right)^{(1/m)}} = \frac{1}{\left(\prod_{i=1}^{m}\sigma_i^2\right)^{(1/m)}}. \tag{2}$$

Denote $\Theta_k = \text{Toep}(e_{k+1})$, $k = 0:N$, where e_k is the unit vector of index k. The orthogonality conditions that the filter bank must satisfy may be written compactly as

$$H^T \Theta_{mk} H = \delta_k I, \quad k = 0 : n. \tag{3}$$

The problem is to maximize $G_c(H)$ from (2) with the constraints (3). Up to now, there is no algorithm to find an optimal solution to this problem. The algorithm proposed in [1] uses a very intuitive heuristic. First find the filter h_1 maximizing σ_1^2 subject to the constraints from (3) on h_1 only. With the obtained h_1, find h_2 maximizing σ_2^2, etc. It turns out that after optimizing h_1, there are few degrees of freedom remaining, and thus the other channels filters are found via an eigenvalue decomposition.

2. THE OPTIMUM COMPACTION FILTER

The problem of finding the optimum compaction filter h_1 for a input signal with given autocorrelation sequence is

$$\text{(H)} \qquad \begin{array}{rll} \rho^o = & \max & h_1^T R h_1 \\ & \text{s.t.} & h_1^T \Theta_{mk} h_1 = \delta_k, \ k = 0 : n, \end{array}$$

where ρ^o is the optimum compaction gain.

Several approaches to this problem have been proposed in the latest years. Moulin *et al* [4] use semi-infinite programming, working on the product filter $G_1(z) = H_1(z)H_1(z^{-1}) = \sum_{k=-N}^{N} g_1(k)z^{-k}$, to find a slightly sub-optimal solution. A fast, but not general, algorithm is given in [5]. An SDP formulation, again using the product filter, has been proposed by Tuqan and Vaidyanathan [6], which provides the optimum filter, but has a very high complexity. In a recent paper [2], the current authors have devised a low complexity SDP method, in the specific form of an eigenvalue minimization. We give here only the basic result.

The problem (H) has the same optimum value as the following SDP problem

$$\text{(D)} \qquad \begin{array}{rll} \lambda^o = & \min & \lambda \\ & \text{s.t.} & X = \lambda I - R + \sum_{k=1}^{n} \mu_k \Theta_{mk} \geq 0 \end{array}$$

Here, the variables are λ and μ_k, $k = 1 : n$. The linear matrix inequality in (D) involves the matrix X, which is Toeplitz and has the diagonals whose indices multiple of m (including the main diagonal) are free, while the other diagonals k are fixed to the values $-r(k)$ of the autocorrelation sequence of the input signal. Hence, the problem (D) consists of the minimization of the maximum eigenvalue λ of a parameterized Toeplitz matrix.

The solution of (D) may be found with very performant primal-dual interior point algorithms. The product filter $G_1(z)$ may be easily retrieved and the optimum compaction filter results via spectral factorization. Usually, the optimum product filter is unique, but there are several spectral factors $H_1(z)$; the choice of such a factor is of no importance for the compaction gain. However, as we will see, the coding gain is affected.

3. FILTER BANK COMPLETION

Once available the optimum compaction filter h_1 for the first channel of the filter bank, we aim now to optimize the filters for the remaining $m-1$ channels. This is the main original contribution of this paper.

The polyphase decomposition of $H_1(z)$ is

$$H_1(z) = \sum_{i=1}^{m} z^{-(i-1)} H_{1i}(z^m). \tag{4}$$

The associated polyphase matrix is

$$E_1(z) = [H_{11}(z) \ H_{12}(z) \ \ldots \ H_{1m}(z)]. \tag{5}$$

Since $H_1(z)$ is on the first channel of a paraunitary filter bank, the orthogonality constraints from (H) are equivalent to

$$E_1(z)E_1^T(z^{-1}) = 1. \tag{6}$$

We will describe in the following the filter bank completion algorithm. For each step, we will describe the operations to be performed and outline their theoretical support.

Step 1. We associate with $E_1(z)$, viewed as the transfer matrix of a linear system with m inputs and one output, a state space representation. The most convenient is the observable one, i.e.

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ 0 & 0 & \ldots & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} h_1(m) & h_1(m+1) & \ldots & h_1(2m-1) \\ h_1(2m) & h_1(2m+1) & \ldots & h_1(3m-1) \\ \vdots & \vdots & & \vdots \\ h_1(nm) & h_1(nm+1) & \ldots & h_1(N) \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} h_1(0) & h_1(1) & \ldots & h_1(m-1) \end{bmatrix} \tag{7}$$

where the shift matrix A has order n.

We compute the solution P of the discrete matrix Lyapunov equation

$$P = APA^T + BB^T, \tag{8}$$

which is positive definite, as the matrix A is stable and the pair (A, B) is controllable (the last row of B is nonzero). Moreover, as (C_1, A) is observable by construction, the orthogonality of the filter $H_1(z)$ is equivalent to the equalities

$$APC_1^T + BD_1^T = 0, \quad CPC^T + DD^T = 1. \tag{9}$$

We compute the Cholesky decomposition of the matrix P, i.e. $L \in \mathcal{R}^{n \times n}$ upper triangular, such that $LL^T = P$.

Step 2. Next, we compute the input balanced state space representation of the polyphase matrix $E_1(z)$. For simplicity, we denote the new matrices with the same letters, i.e.

$$A \leftarrow L^{-1}AL, \quad B \leftarrow L^{-1}B, \quad C_1 \leftarrow C_1L. \tag{10}$$

In this form, the paraunitarity of $E_1(z)$ is expressed as the orthogonality of the rows of the $(n+1) \times (n+m)$ system matrix

$$S_1 = \left[\begin{array}{cc} A & B \\ C_1 & D_1 \end{array} \right], \tag{11}$$

i.e. $S_1 S_1^T = I$, see (8), (9).

Step 3. Now, the matrix S_1 is completed with $m-1$ rows, such that an orthogonal matrix

$$S = \left[\begin{array}{cc} A & B \\ C & D \end{array} \right] \tag{12}$$

of size $(n+m) \times (n+m)$ is obtained, where

$$C = \left[\begin{array}{c} C_1 \\ C_2 \end{array} \right], \quad D = \left[\begin{array}{c} D_1 \\ D_2 \end{array} \right]. \tag{13}$$

The orthogonal completion may be computed in a numerically stable manner by performing the QR factorization of S_1^T.

Notice that once an orthogonal completion C_2, D_2 has been found, for any orthogonal matrix U_2 of size $(m-1) \times (m-1)$, the matrices U_2C_2, U_2D_2 are also a valid completion. Also, all completions may be computed in this way. We will show later how to choose U_2.

Step 4. The state space representation (A, B, C, D) belongs to a system with m inputs and m outputs, whose transfer matrix is paraunitary and also may be seen as the polyphase matrix of an m-channel filter bank with $H_1(z)$ on the first channel

$$E(z) = \left[\begin{array}{cccc} H_{11}(z) & H_{12}(z) & \dots & H_{1m}(z) \\ \vdots & \vdots & & \vdots \\ H_{m1}(z) & H_{m2}(z) & \dots & H_{mm}(z) \end{array} \right] \tag{14}$$

The coefficients of the remaining filters are determined by computing the Markov coefficients D, CB, CAB, ..., $CA^{n-1}B$, since

$$E(z) = D + \sum_{i=1}^{n} z^{-i} CA^{i-1}B. \tag{15}$$

Step 5. We now compute the orthogonal matrix U_2 in order to maximize the coding gain. Using H from (1), let compute

$$\tilde{R} = H^T RH. \tag{16}$$

Proposition 1. The matrix $\tilde{R}$ has the structure

$$\tilde{R} = \left[\begin{array}{cc} \rho^o & 0 \\ 0 & R_2 \end{array} \right]. \tag{17}$$

Proof. From the optimality of the compaction filter h_1 and the remark that if H satisfy the constraints (3), then HV also satisfies them, for any orthogonal matrix V. ■

Proposition 2. The orthogonal matrix U_2 maximizing the coding gain is given by the eigendecomposition

$$U_2^T R_2 U_2 = \Lambda, \tag{18}$$

where Λ is a diagonal matrix (with positive diagonal elements). The filter bank is finally given by

$$H \left[\begin{array}{cc} 1 & 0 \\ 0 & U_2 \end{array} \right], \tag{19}$$

and, similarly to H, contains the filters coefficients on its columns.

Proof. If $m-1=2$, then the coding gain is maximized when $U_2^T R_2 U_2$ is diagonal (if the sum of two positive numbers is constant —the trace of R_2— then their product is minimum when the numbers reach their extreme values —the eigenvalues of R_2).

If $m-1>2$, the same argument may be applied to the Jacobi algorithm for diagonalizing a symmetric matrix, using 2×2 rotations that succesively diagonalize 2×2 principal submatrices of R_2 (and the resulting similarity transformations are accumulated in R_2). ■

The overall complexity of this algorithm is $O((n+m)^3)$ operations, similar to that of the algorithm of Moulin and Mihçak [1]. Anyway, the complexity is far lower than for finding the optimum compaction filter.

m	$N+1$	Input	Coding gain			
			Best	Worst	Min.ph.	Max.ph.
3	24	AR(1)	7.736	7.600	7.678	7.688
		AR(2)	5.164	3.021	3.021	4.401
		AR(8)	8.295	6.570	6.570	7.572
4	24	AR(1)	8.493	8.312	8.383	8.426
		AR(2)	8.868	8.519	8.667	8.868
		AR(8)	6.026	5.385	5.944	5.385
8	48	AR(1)	9.430	9.217	9.254	9.361
		AR(2)	10.023	9.838	9.838	10.023
		AR(8)	8.672	7.877	7.877	8.672

Table 1: Coding gains for different phase choices.

We have not made an extensive comparison, but we appreciate that our algorithm has better numerical properties.

Considerations on the optimality. Proposition 2 shows how to maximize the coding gain when the filter in the first channel is fixed. As we will show experimentally in the next section, the different filters obtained from the spectral factorization, although giving the same compaction gain, lead to filter banks with significantly different coding gains. Moreover, as shown in [3], maximizing the coding gain and the compaction gain have not the same solution, although usually the differences are minor.

The filter banks resulted from the algorithm above satisfy, additionally to the orthogonality constraints (3), the property that the matrix H^TRH is diagonal. Similarly to Proposition 2, it may be proved immediately that the filter bank with *optimum coding gain* satisfies the same property.

4. NUMERICAL RESULTS

For the computation of the optimum compaction filter, we used the SDPT3 package of Toh, Todd and Tütüncü [7], designed for general SDP problems and written in Matlab. For the completion of the filter bank, we have written a dedicated program, also in Matlab.

The test autocorrelation sequences were generated using autoregressive models, namely

- AR(1) with the pole $\rho = 0.95$.
- AR(2) with poles at $\rho e^{\pm j\theta}$, with $\rho = 0.975$ and $\theta = \pi/3$.
- AR(8) with two double pairs of poles with $\rho = 0.9$ and $\theta = \pi/4$ and $3\pi/5$, respectively.

Accuracy. After computing a filter bank H with our algorithm, we computed the error in satisfying the orthogonality constraints (3). This error is typically of the order 10^{-12} for $N+1 = 100$ or even smaller for smaller N, which shows excellent numerical properties.

Experimental coding gains. We first compared our results with those from [1], where the minimum phase filter $H_1(z)$ is chosen at the spectral factorization. Our coding gains are slightly better, usually in the second or third decimal digit; the cause is the optimality of our compaction filter, while the filter in [1] is suboptimal.

Then, we evaluated the effect of the choice of the spectral factor. To this purpose, we computed the coding gain for all possible choices of the phase of the optimum compaction filter $H_1(z)$. The results are presented in Table 1. They show that the coding gain may vary significantly with the choice of the phase. Apparently, choosing the maximum phase yields better coding gains, while the minimum phase is often the worst. However, we discovered no general rule to guide the choice of the phase.

5. REFERENCES

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