On the stability of active filters

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Abstract

The general relationship between the stability behavior of active filters for ideal and "real" integrators is investigated using Cauchy's theorem. The results can be used for the comparative analysis of several configurations of continuous-time state variable active filter with "real" integrators based on operational amplifiers ($OPAMP$'s), transconductance (OTA's) and transresistance (ORA's) amplifiers.

1 Introduction

Due to the presence of active integrator devices, active filter design implies first of all stability conditions fulfillment. Accurately modeling of such devices is fundamental for finding the range of the parameters for stable functioning [1,2]. Even though thorough computer simulations can be used as a (necessary) "brute force" design tool, analytical guidelines are always welcome. However, any analytical result will be more or less valuable, according to the tradeoff between simplicity and accuracy. The active integrator models can range from the elementary $\frac{1}{s}$ transfer function (corresponding to ideal $OPAMP$'s, OTA's or ORA's) to high order two-port nonlinear models that are almost impossible to be analytically handled. A reasonable model for the integrators, consider them as linear devices having finite d.c. gain and one or several poles. These models are suited for stability analysis by means of the powerful methods of linear systems theory.

Such an approach, based on the Routh-Hurwitz criterion, has been used in [3,4] for the stability study of nonideal $OPAMP$, $OTA-C$ and $ORA-L$ integrator active filter configurations. The investigations envisaged two pole identical active integrators biquad structures and used a mild approximation within the Routh-Hurwitz test.

The aim of this paper is to further refine the above stability results using Cauchy's theorem for a general nonideal integrator biquad transfer function based on second and third order identical integrators. The method can be extended for higher order integrators and/or transfer functions but in this paper only biquad structures will be considered. This unified approach gives the tools for stability analysis of all active filters configurations.

In the following the transfer and transmission functions of the integrators will be denoted by $H_{int}(s)$ and $T_{int}(s)$ respectively ($T_{int}(s) = 1/H_{int}(s)$).

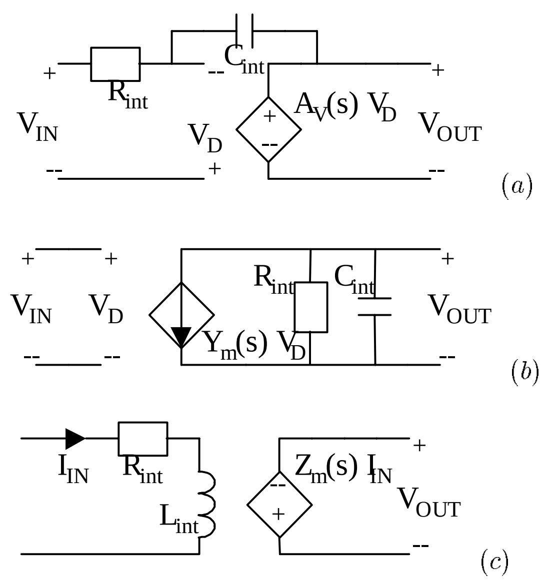
In the ideal case, these functions are given by:

$$H_{int_0}(s) = \frac{1}{T_{int_0}(s)} = \frac{\omega_{int}}{s} \tag{1}$$

In the next section, the general case including nonideal integrator structures based on linear nonideal $OPAMP$'s, OTA's, ORA's will be considered.

2 Nonideal integrator transmission and transfer functions

In view of a unified approach, the transresistance and transconductance amplifiers will be transformed into voltage amplifiers. An active integrator structure includes an amplifier and several passive elements connected either in a feedback topology ($OPAMP$) or a feedforward one (OTA, ORA). The amplifiers are modeled as frequency dependent controlled sources as shown in $Fig.1$



$Fig.1$ Nonideal integrators models based on $OPAMP$ (a), OTA (b), ORA (c)

The following considerations will be based on the observation that the formulas of the above integrators transfer functions can be cast in one of two forms:

$$H_{intFB}(s) = \frac{1}{T_{intFB}(s)} = \frac{-A(s)B(s)}{1 + A(s)[1 - B(s)]} \tag{2}$$

$$H_{intFF}(s) = \frac{1}{T_{intFF}(s)} = -A(s)B(s) \tag{3}$$

OTA/ORA) integrator topologies respectively.

In (2) and (3), $A(s)$ has the significance reflected by the relations (4), (5) and (6) for $OPAMP$, OTA and ORA, i.e.:

$$A(s) = A_V(s) \quad (4)$$
$$A(s) = Y_m(s) R_{int} \quad (5)$$
$$A(s) = Z_m(s)/R_{int} \quad (6)$$

while $B(s)$ is:

$$B(s) = \frac{\omega_{int}}{s+\omega_{int}} \quad (7)$$

with ω_{int} :

$$\omega_{int} = 1/(R_{int}C_{int}) \quad (8)$$
$$\omega_{int} = 1/(R_{int}C_{int}) \quad (9)$$
$$\omega_{int} = L_{int}/R_{int} \quad (10)$$

for the three cases respectively.

The stability study below considers noninverting integrators; the inverting case can be handled similarly or converted to the noninverting case through the sign of the feedback coefficients.

3 Stability of active biquads

The following considerations will concern the stability of the general biquad configuration where the (nonideal) integrators can be built using any of the three structures ($FB-OPAMP$, $FF-OTA, ORA$) discussed above.

The transfer function of the biquad is:

$$H_{biquad}(s) = k\frac{s^2+b_1s+b_2}{s^2+a_1s+a_2} \quad (11)$$

In the nonideal case the complex variable s should be replaced by T_{int}. Thus, the stability of the nonideal biquad is determined by the characteristic polynomial:

$$Q(s) = T_{int}^2(s) + a_1T_{int}(s) + a_2 \quad (12)$$

which, for the ideal case of the prototype ideal biquad, obviously takes the simple form:

$$Q_p(s) = s^2 + a_1s + a_2 \quad (13)$$

The following considerations will be based on the factorization of the prototype characteristic polynomial:

$$Q_p(s) = (s-p_1)(s-p_2) \quad (14)$$

and the corresponding factorization of the "nonideal" characteristic polynomial:

$$Q(s) = (T_{int}(s)-p_1)(T_{int}(s)-p_2) \quad (15)$$

a)The factorization (15) is valid only if the two integrators are identical.

b) Two sets of parameters can be adjusted independently to ensure the filter stability:

1)the resistances which determine the coefficients a_1, a_2 and thus, the roots p_1, p_2.

2)the integrator parameters (resistances, capacitances, inductances), which define the transmission function $T_{int}(s)$.

As $T_{int}(s)$ has no poles, $T_{int}(s)-p_k$ is a polynomial function; it will be supposed to have Z zeros in the right half plane. The stability study is based on the application of Cauchy's theorem for each factor of the form $T_{int}(s)-p_k$ from (15) for a Nyquist contour C_N (consisting of the imaginary axis and a semicircle with infinite radius placed in the right half plane) where p_k are the roots of the prototype characteristic polynomial $Q_p(s)$. This contour obviously contains all zeros of $T_{int}(s)-p_k$ placed in the right half plane. According to Cauchy's theorem, the hodograph of $T_{int}(s)-p_k$, i.e. $H = \{T_{int}(s)-p_k\}$ for s belonging to the Nyquist contour, will circle clockwise the origin Z times i.e., the argument variation of $T_{int}(s)-p_k$ will be $2\pi Z$. Equivalently, the hodograph of $T_{int}(s)$ will circle clockwise the point p_k Z times. Thus, for fixed parameters of the nonideal integrators, the stability of the transfer function $1/(T_{int}(s)-p_k)$ is ensured if the p_k values are chosen such that $\Delta\arg[T_{int}(s)-p_k] = 0$, i.e. $Z=0$. Our concern is to find the stability domains in the complex plane for p_k, the roots of the prototype characteristic polynomial.

The stability problem will be solved by tracing the hodograph of the function $T_{int}(s)$ for s belonging to the Nyquist contour and looking for the points in the complex plane not circled by the hodograph.. Any of these points represent an admissible position of p_k for stability.

We denote by $x(\omega)$ and $y(\omega)$ the real and imaginary parts of $T_{int}(s)$ for $s=j\omega$:

$$\begin{cases} x(\omega) = \mathrm{Re}\,T_{int}(j\omega) \\ y(\omega) = \mathrm{Im}\,T_{int}(j\omega) \end{cases} \quad (16)$$

3.1 One pole amplifiers/two poles integrators

In the case of a one-pole amplifier transfer function of the form

$$A_1(s) = \frac{a_0\omega_{p_1}}{s+\omega_{p_1}} \quad (17)$$

using the amplifier in a feedback structure ($OPAMP$) or in a feedforward structure (OTA, ORA), from the relations(2), (3), (7) and (17), the following (noninverting) integrator transmission functions are obtained respectively:

$$T_{intFB1}(s) = \frac{(s+\omega_{p_1})(s+\omega_{int})}{a_0\omega_{p_1}\omega_{int}} + \frac{s}{\omega_{int}} \quad (18)$$

$$a_0\omega_{p_1}\omega_{int}$$

The polar diagram of the integrator transmission function in each case is described by two parametric equations which define a parabola:

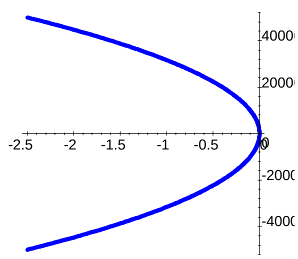
$$\begin{cases} x_{FB1}(\omega) = \frac{-\omega^2+\omega_{p_1}\omega_{int}}{a_0\omega_{p_1}\omega_{int}} \\ y_{FB1}(\omega) = \frac{\omega\left(\omega_{p_1}(1+a_0)+\omega_{int}\right)}{a_0\omega_{p_1}\omega_{int}} \end{cases} \quad (20)$$

$$\begin{cases} x_{FF1}(\omega) = \frac{-\omega^2+\omega_{p_1}\omega_{int}}{a_0\omega_{p_1}\omega_{int}} \\ y_{FF1}(\omega) = \frac{\omega\left(\omega_{p_1}+\omega_{int}\right)}{a_0\omega_{p_1}\omega_{int}} \end{cases} \quad (21)$$

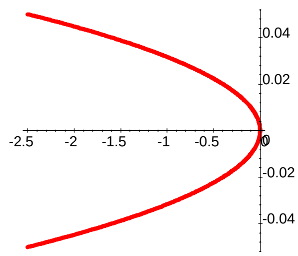
where $x_{FB1}(\omega)$, $y_{FB1}(\omega)$ and $x_{FF1}(\omega)$, $y_{FF1}(\omega)$ have the significance from equations (16).

For fixed values a_0, ω_{p1}, ω_{int} of the integrator parameters the parabola has the shape shown in $Fig.2$ $a)$ and $b)$ for FB and FF structures respectively:

$a_0 = 10^6, \omega_{p_1} = 10^9, \omega_{int} = 10^6$



$Fig.2a)$



$Fig.2b)$

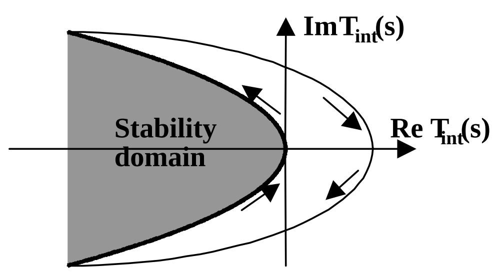


Fig.3 Hodograph of $T_{int}(s), s \in C_N$

The shape of the global hodograph in both cases is sketched in $Fig.3$. Thus, the domain of stability for p_k is the domain inside the parabola as the above points are not circled by the hodograph; the whole parabola represents the stability frontier. Based on that, we can deduce the global analytic expression of the stability conditions for p_k complex conjugate (22) or real (23), respectively:

$$\operatorname{Re} p_k \le \alpha_0 - C \operatorname{Im}^2 p_k \quad (22)$$

$$p_k \le \alpha_0 \quad (23)$$

where C represents the parabola's curvature and should be replaced by C_{FB} and C_{FF} for the FB and FF apex. These parameters are given below:

$$\alpha_0 = 1/a_0 \quad (24)$$

$$C_{FB} = \frac{a_0\omega_{p_1}\omega_{int}}{\left(\omega_{p_1}(1+a_0)+\omega_{int}\right)^2} \quad (25)$$

$$C_{FF} = \frac{a_0\omega_{p_1}\omega_{int}}{\left(\omega_{p_1}+\omega_{int}\right)^2} \quad (26)$$

Let us remark that $\operatorname{Re} p_k$ can be positive so that, *even though the (ideal) prototype filter is unstable, the use of nonideal integrators can make it stable.*

If we compare the stability domains of both types of structures from figures 2 $a)$ and $b)$, we observe that in the first case (FB) the stability domain is larger than in the second case (FF) , but this is a formal result as we refer to different types of integrators, which may have very different values of parameters (for example, the bandwidth of an OTA or ORA is comparable with the product gain-bandwidth of an $OPAMP$).

3.2 Two poles amplifiers/three poles integrators

In the case of an amplifier having a two-poles transfer function of the form:

$$A_2(s) = \frac{a_0\omega_{p_1}\omega_{p_2}}{(s+\omega_{p_1})(s+\omega_{p_2})} \quad (27)$$

the (noninverting) integrator transmission functions for the FB and FF configurations are respectively:

$$T_{intFB2}(s) = \frac{(s+\omega_{p_1})(s+\omega_{p_2})(s+\omega_{int})}{a_0\omega_{p_1}\omega_{p_2}\omega_{int}} + \frac{s}{\omega_{int}} \quad (28)$$

$$T_{intFF2}(s) = \frac{(s+\omega_{p_1})(s+\omega_{p_2})(s+\omega_{int})}{a_0\omega_{p_1}\omega_{p_2}\omega_{int}} \quad (29)$$

As in the previous section, the polar diagram of the integrator transmission function in both cases is described by two parametric equations:

$$\begin{cases} x_{FB2}(\omega) = \frac{-\omega^2\left(\omega_{p_1}+\omega_{p_2}+\omega_{int}\right)+\omega_{p_1}\omega_{p_2}\omega_{int}}{a_0\omega_{p_1}\omega_{p_2}\omega_{int}} \\ y_{FB2}(\omega) = \frac{-\omega^3+\omega\left(\omega_{p_1}\omega_{p_2}(1+a_0)+\omega_{int}\omega_{p_1}+\omega_{p_2}\omega_{int}\right)}{a_0\omega_{p_1}\omega_{p_2}\omega_{int}} \end{cases} \quad (30)$$

$$\begin{cases} x_{FF2}(\omega) = \frac{-\omega^2\left(\omega_{p_1}+\omega_{p_2}+\omega_{int}\right)+\omega_{p_1}\omega_{p_2}\omega_{int}}{a_0\omega_{p_1}\omega_{p_2}\omega_{int}} \\ y_{FF2}(\omega) = \frac{-\omega^3+\omega\left(\omega_{p_1}\omega_{p_2}+\omega_{int}\omega_{p_1}+\omega_{p_2}\omega_{int}\right)}{a_0\omega_{p_1}\omega_{p_2}\omega_{int}} \end{cases} \quad (31)$$

The polar diagrams of the function $T_{int}(j\omega)$ for the same values of a_0, ω_{p_1}, ω_{int} fixed in the previous section and an ω_{p_2} with three decades greater than ω_{p_1} are plotted in $Fig.4$ $a)$ and $b)$ for FB and FF structure, respectively:

$a_0 = 10^6, \omega_{p_1} = 10^9, \omega_{p_2} = 10^{12}, \omega_{int} = 10^6$

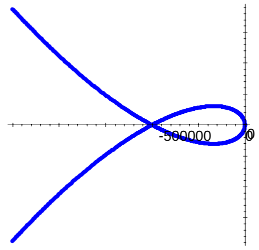


Fig.4a)

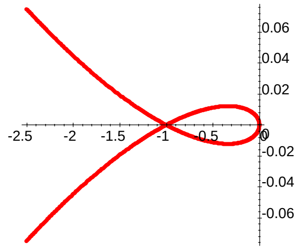


Fig.4b)

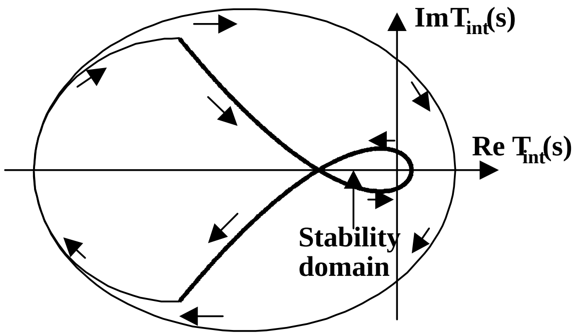


Fig.5 Hodograph of $T_{int}(s), s \in C_N$

According to the global polar diagram sketched in *Fig.*5, it is obvious that the domain of stability is now the area inside the loop from the polar diagram of $T_{int}(j\omega)$.

The parametric equations of the stability frontiers are the same as the parametric equations of the integrator transmission function polar diagram with the constraint:

$$-\omega_x \leq \omega \leq +\omega_x \tag{32}$$

where ω_x is either ω_{xFB} or ω_{xFF} and represent the positive frequency where the polar diagram of $T_{int}(j\omega)$ crosses the x axis. With some algebra (details will be given in another paper), the general analytic expressions of the stability conditions have the form (33) if p_k are complex conjugate or (34) if p_k are real :

$$\begin{cases} |\mathrm{Im}\, p_k| \leq \alpha_0 \dfrac{\omega_y \sqrt{1-\frac{\mathrm{Re}\, p_k}{\alpha_0}} \left[\omega_x^2 - \omega_y^2 \left(1-\frac{\mathrm{Re}\, p_k}{\alpha_0}\right)\right]}{\omega_g^3} \\ \mathrm{Re}\, p_k \in \left[\alpha_0 \left(1-\frac{\omega_x^2}{\omega_y^2}\right), \alpha_0\right] \end{cases} \tag{33}$$

$$p_k \in \left[\alpha_0 \left(1-\frac{\omega_x^2}{\omega_y^2}\right), \alpha_0\right] \tag{34}$$

These relations give the stability conditions for the FB and FF cases by replacing ω_x, ω_y with ω_{xFB}, ω_{yFB} and ω_{xFF}, ω_{yFF}, respectively, taken from the expressions given below:

$$\alpha_0 = 1/a_0 \tag{35}$$

$$\omega_g = \sqrt[3]{\omega_{p_1}\omega_{p_2}\omega_{int}} \tag{36}$$

$$\omega_{yFB} = \sqrt{\frac{\omega_{p_1}\omega_{p_2}\omega_{int}}{\omega_{p_1}+\omega_{p_2}+\omega_{int}}} \tag{38}$$

$$\omega_{xFF} = \sqrt{\omega_{p_1}\omega_{p_2}+\omega_{p_1}\omega_{int}+\omega_{p_2}\omega_{int}} \tag{39}$$

$$\omega_{yFF} = \sqrt{\frac{\omega_{p_1}\omega_{p_2}\omega_{int}}{\omega_{p_1}+\omega_{p_2}+\omega_{int}}} \tag{40}$$

Again, *it is possible to obtain a stable filter from an unstable prototype.*

We observe from the figures 4 *a*) and *b*) that in the case of the feedback structure, the stability domain is larger than in the case of the feedforward structure, like in the previous section.

4 Concluding remarks

The stability of a continuous-time state variable active filters built with nonideal $OPAMP$, ORA and OTA based integrators has been studied using Cauchy's theorem. The stability conditions show the possibilities of an independent adjustment of two sets of parameters:

a)the resistances of the linear network which interconnect the integrator cells

b)the integrator parameters

Thus, the conditions the interconnecting network parameters should satisfy, for given integrator (identical) cells, to ensure the stability of the filter have been determined. The study has been done for second order and third order integrators and can be generalized, in principle for any order, provided there are no zeros in the integrators models.

Further investigations should consider the input and output amplifiers impedance influences on the stability conditions.

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