

MODIFIED B-SPLINE FUNCTIONS FOR EFFICIENT IMAGE INTERPOLATION

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ABSTRACT

A parametric class of modified B-spline functions is studied for its applicability in digital image interpolation tasks. The modified functions are generated by weighting and shifting of classical B-splines of different degrees. The weighting coefficients are optimized to assure an effective attenuation of the imaging frequencies, caused by the interpolation. The most computationally expensive part of the interpolation procedure – generation of new pixels at arbitrary points, is realized efficiently using a modified Farrow structure. Different interpolation methods are experimentally compared for their capability to perform successive rotations. The new modified B-spline interpolating kernels show superior performance while the computational complexity remains practically the same.

1. INTRODUCTION

The interpolation makes the relationship between the discrete and the continuous-time world. Usually samples at discrete points are presented only, and one has to fit certain continuous-time function with them in order to get the function values at intermediate points. Concerning the discrete images, the interpolation is involved in such tasks as image magnification (enlargement), image rotation, resizing, etc. In the general case the interpolation factor is arbitrary, not necessary integer, even not rational.

The simplest interpolation method is the nearest neighbor, when the interpolating kernel is representing by a box function. While it is very easy to realize, it leads to low quality in the most applications. The second limit case is the sinc ($\sin(x)/x$) interpolator. It can be chosen assuming that the discrete values represent any band-limited function sampled in accordance with the Shannon theory. While the sinc function theoretically is the ideal interpolator, it is very hard to realize in practice because of its infinite support and slow delay [1].

An alternative assumption is that the continuous model belongs to some polynomial spline space. This gives some facilitation, especially if the splines are represented in terms of B-spline basic functions [2]. The nicest properties of B-splines are their compact support, sufficient regularity, smooth behavior, and easy implementation [2]. Several attempts to increase the spline interpolation performance have been done by modifying the basic functions [3], [4]. In general, they lead to linear combinations of weighted and shifted classic B-splines and a question arises how to choose the weighting parameters.

This paper is aimed at studying several B-spline modifications for their applicability to the digital image interpolation. A technique for optimizing the modification parameters is presented together with an efficient realization based on a modified Farrow structure. The paper is organized as follows. In Section 2 we represent both classical and modified B-spline basic functions. In Section 3 we then formulate the B-spline interpolation model consisting of two main steps. Efficient realizations of these two steps are discussed. Special attention is paid on the calculation of the interpolating B-spline values by a modified Farrow structure. Finally, we summarize with the experimental results and conclusions.

2. B-SPLINE BASIC FUNCTIONS

2.1 Classical B-spline basis functions

The normalized B-spline functions of degree n with $n+2$ equally spaced data points at the integers are defined by

$$\beta^n(x) = \sum \frac{(-1)^i}{n!} \binom{n+1}{i} (x-i)^n_+ u(x-i), \quad (1)$$

where $u(x)$ is the unit step [2]. They belong to the class C^{n-1} , which means that they are continuous and have $n-1$ continuous derivatives. The B-spline functions can be obtained recursively by convolutions starting from the B-spline of degree zero:

$$\beta^0(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \quad (2)$$

and

$$\beta^n = \beta^0 * \beta^{n-1}. \quad (3)$$

The corresponding frequency-domain characteristics are defined as follows:

$$B^n(\omega) = \left(\frac{\sin(\omega/2)}{(\omega/2)} \right)^{n+1} \quad (4)$$

The integer shifts of the B-spline function of degree n form a Riesz basis for the spline space $V(\beta^n)$, i.e. all polynomial splines of degree n can be represented as linear combination of B-spline basic functions of the same degree

$$s^n(x) = \sum_{k \in \mathbb{Z}} c(k) \beta^n(x-k). \quad (5)$$

In fact, Eq. (5) represents a 1-D interpolation model. It can be easily extended for multidimensional cases by applying a tensor product of separable 1-D bases.

The B-splines possess a number of nice properties: they are compactly supported on the interval $[0, n+2]$, having sufficient regularity and smooth behavior. They satisfy the so-called partition of unity condition [1], [5] which is obligatory for the interpolation kernels. Their approximation capability is measured by the so-called approximation order $L=n+1$. The B-spline of degree n and approximation order L can represent exactly polynomials up to L -th order. It has been proved that the B-splines of 3rd degree ($L=4$) assure good compromise between the interpolated signal's quality and the computational complexity. Higher interpolation quality is accessible for higher degree splines giving up the lower computational complexity.

2.2 Modified B-spline basis functions

The following question arises: can one improve the approximation properties of the B-spline basic function at the expense of a negligible additional computational cost. Analyzing Eq. (5) one can realize that the computational complexity is related with the basic functions support. Therefore, it is meaningful to preserve the support (and the approximation order as well) up to any number N by the following B-spline basic function modification:

$$\beta^{\text{mod}}(x) = \sum_{n=0}^N \sum_{m \in \mathbb{Z}} \gamma_{nm} \beta^n(x-m) \quad (6)$$

The function β^{mod} is the generating function for some spline space $V(\beta^{\text{mod}})$ where each function which belongs to this space can be represented by Eq. (5) formally replacing β^n by β^{mod} . The weighting coefficients 'gammas' provide more degrees of freedom giving us the potential of tuning the modified B-spline approximation properties in an application-oriented manner.

All suitable combinations in Eq. (6) form a class of functions called by Thevenaz *et al.* [5] Moms –'maximal (approximation) order - minimum support' which, in fact coincides with earlier proposed modified B-spline interpolators [3]. Actually, Blu *et al.* [4] have combined a central B-spline of degree n with its first $n-1$ derivatives. Since a derivative of a B-spline is a combination of two shifted by half B-splines of lower degree [5], their construction leads to a form very close to (6). They have optimized the weighting coefficients to minimize the asymptotic approximation error. The resulting interpolating functions are especially suitable for low-frequency images [5]. Blu *et al.* [4] have obtained optimized coefficients for the linear combination of B-splines of 5th, 3rd and 1st degrees (5th degree spline and its even derivatives) and for the combination of 3rd and 1st degree B-splines (3rd degree B-spline and its second derivative).

Our optimization approach is based on the assumption that we have to suppress effectively the imaging frequencies, generated by the interpolation. In this contribution we concentrate on generating modified B-spline functions using 3rd and 1st degree splines involved, although other modifications are also possible.

The weighting coefficients optimization technique will be explained further in the following sections.

3. B-SPLINE INTERPOLATION MODEL

In the classical interpolation task one aims at fitting a continuous-time function with the input discrete data and then resampling this function by an arbitrary (finer) sampling step. The continuous-time function values at the integer points must coincide with the discrete sequence values. Combining this interpolation constraint with the model of Eq. (5) we obtain the interpolation scheme shown in Fig. 1.

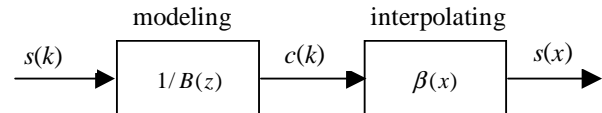


Figure 1. Block diagram of the B-spline interpolation model.

3.1 Modeling step

At the integer points Eq. (5) takes the following form:

$$s(x)|_{x=l} = s(l) = \sum_{k \in \mathbb{Z}} c(k) \beta(l-k) \quad (7)$$

It represents a convolution between discrete sequences and can be written in z -domain as follows:

$$S(z) = C(z)B(z) \quad (8)$$

The first step in the B-spline interpolation scheme is an inverse filtering giving the spline model coefficients $c(k)$. The inverse filter $1/B(z)$ is formed by sampling the B-spline function at the integers [6]. For symmetrical B-spline functions (classical or modified) it is a symmetrical IIR filter which can be realized by successive forward and backward IIR filters of the first order [6] or by an efficient FIR approximation [3]. The regular and the modified cases do not differ because the practical realization of Eq. (8) depends on the number of roots of the polynomial $B(z)$ which is equal to the highest spline degree $N-1$ in Eq. (6). Hence, we can model image pixels as a spline function with acceptable additional computations in comparison with the traditional interpolation, thus taking advantage of the properties of B-spline and modified B-spline basic functions.

3.2 Interpolating step

Grace to the compact support of the basis functions Eqs. (1) and (6) only a finite number of coefficients take part in the interpolating step given by Eq. (5) for a particular point μ . For example for a B-spline of 3rd degree there are four basic functions involved in calculation of the function value at any point μ . Although Eq. (1) can represent the B-spline function at an arbitrary point, it remains the most computationally expensive part in realization of the interpolation formula (5). Instead of direct application of (1) we apply here a modification of the well-known Farrow structure for the efficient realization of Eq. (5). The Farrow structure has been used to form a

piecewise polynomial approximation to the sampled data and to evaluate this polynomial at the desired (arbitrary) point [7]. It consists of $n+1$ parallel FIR branch filters, where n is the degree of the polynomial [8]. Here we use the branch filters of the Farrow structure to form a modified B-spline approximation of the image data. We start with rewriting the interpolation formula (5) for the fractional interval, changing the variable x by μ :

$$\mu = x - m. \quad (9)$$

Then Eq. (5) becomes

$$s(\mu) = \sum_{k \in \mathbb{Z}} c(k) \beta^{\text{mod}}(\mu + m - k), \quad (10)$$

where

$$\beta^{\text{mod}}(x) = \beta^3(x) + \gamma_{10} \beta^1(x) + \gamma_{11} \beta^1(x-1) + \gamma_{11} \beta^1(x+1) \quad (11)$$

for the 3-1 modified case.

Then we represent Eq. (10) in the polynomial form for the interval (0,1] using Eq. (1), yielding

$$\begin{aligned} s(\mu) &= \sum_{k=-2}^1 c_k \beta^{\text{mod}}(\mu - k + 1) = \\ &= \frac{1}{6} c_{-2} [(\mu + 3)^3 - 4(\mu + 2)^3 + 6(\mu + 1)^3 - 4\mu^3] + \\ &+ \frac{1}{6} c_{-1} [(\mu + 2)^3 - 4(\mu + 1)^3 + 6\mu^3] + \\ &+ \frac{1}{6} c_0 [(\mu + 1)^3 - 4\mu^3] + \frac{1}{6} c_1 \mu^3 + \\ &+ \gamma_{00} c_{-2} [(\mu + 2) - 2(\mu + 1) + \mu] + \\ &+ \gamma_{00} c_{-1} [(\mu + 1) - 2\mu] + \gamma_{00} c_0 \mu + \\ &+ \gamma_{10} c_{-2} [(\mu + 3) - 2(\mu + 2) + (\mu + 1)] + \\ &+ \gamma_{11} c_{-1} [(\mu + 2) - 2(\mu + 1) + \mu] + \\ &+ \gamma_{11} c_0 [(\mu + 1) - 2\mu] + \gamma_{11} c_1 \mu + \\ &+ \gamma_{11} c_{-2} [(\mu + 1) - 2\mu] + \gamma_{11} c_{-1} \mu \end{aligned} \quad (12)$$

After some rearrangements $s(\mu)$ is expressible using matrix notations as

$$s(\mu) = \frac{1}{6} \begin{bmatrix} \mu^3 & \mu^2 & \mu & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 - 6\gamma_{11} & -6\gamma_{10} + 6\gamma_{11} & 3 + 6\gamma_{10} - 6\gamma_{11} & -6\gamma_{11} \\ 1 + 6\gamma_{11} & 4 + 6\gamma_{10} & 1 + 6\gamma_{11} & 0 \end{bmatrix} \begin{bmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \end{bmatrix} \quad (13)$$

Eq. (13) represents a filter structure consisting of four FIR filters each having fixed coefficients. The interpolant point $s(\mu)$ can be obtained by multiplying the output of the m^{th} FIR filter by μ^{m-1} . The only value which has to be loaded to the interpolator is the fractional interval μ .

3.3 Optimizing the weighting coefficients

The main idea of using a linear combination of the cubic and the linear B-splines is that we can improve the performance of the interpolator without increasing the complexity. Intuitively, cubic splines would be responsible for the smooth behavior while the

linear splines would be responsible for the edges representation. Their combined action can be expressed in the frequency domain as sufficient suppression of the imaging frequencies in the stop-band of the interpolating filter. Hence, our optimization task can be defined as follows:

- Consider the normalized frequency rate $F_s=1$.
- Assume that the most important signal components are in the frequency range $f \leq \alpha F_s/2$, with $\alpha < 1$.
- Let δ_s be the largest amplitude value in those frequency ranges where the images of the most important input signal components exist.
- Let δ_p be the maximum amplitude deviation of the interpolation filter from unity in the range $f \leq \alpha F_s/2$
- Optimize γ_{10} and γ_{11} to minimize $\max(\delta_p/w, \delta_s)$ for the given value of w .
- The final constraint: $\gamma_{10} = -2\gamma_{11}$ is forced by the partition of unity condition [1], [5].

4. EXPERIMENTAL RESULTS

4.1 Rotation through successive translations

How to evaluate different image interpolators? One possibility is to reduce the image by a certain decimation factor and then to enhance it to the original size by interpolation. In order to be perfectly correct, we need to apply a decimation operator being optimal with respect to the interpolator. In the light of the modern sampling theory this means to make an orthogonal projection by the basis dual to the interpolation one. In this case a *mse* and a *SNR* would give an integral measure about the decimator-interpolator performance. An alternative is to use rotation as a task involving interpolation. Such approach was introduced in [9] based on the decomposition of the rotation matrix into three matrices. It leads to three successive translations: through the x axis, through the y axis, and through the x axis again [9]. It does not require any rescaling and can be used as a pure interpolation evaluator.

We have performed ten successive image rotations by an angle of 36° on a number of gray-scale images. We have applied the B-spline interpolation model for the following bases: classical b3 spline, case 3-1 o-Moms [4], and case 3-1 modified spline with different sets of coefficients obtained via the optimization procedure described in subsection 3.3. Table 1 summarizes the results for two 256x256 points images (Baboon1 and Geometrical) and for two 512x512 points images (Barbara and Baboon2). The images were especially chosen to have more high frequency details. SNRs have been computed between the central parts of the initial image and the rotated one. The parameter w represents the constraint forced to limit the admissible ripples in the pass-band region. Fig. 2 and Fig. 3 show typical results for the central part of the Geometrical image and for the right lower corner of the Barbara image, respectively. It can be pointed out the good preservation of the high frequency details for the case of modified B-splines with appropriately adjusted weighting parameters.

5. CONCLUSIONS

A class of modified B-spline basic functions has been studied for its applicability to digital image interpolation. The modification has been made by linear combination of classical B-splines of different degrees. The combination of 3rd and 1st degrees splines has been especially studied because it provides good opportunity to deal with both smooth and sharp image areas and because it offers acceptable computational complexity. An optimization technique, which attenuates the imaging frequencies in the stop-band of the interpolating filter, has been applied. Such modified B-spline kernels can dramatically increase the interpolation quality as seen from the experimental results obtained. An efficient practical realization based on the modified Farrow structure has been introduced. Other possible applications of the recently developed modified B-spline functions e.g. digital image resizing are also very promising and are under research.

6. ACNOWLEDGEMENTS

Authors wish to thank the Median Free Group International for excellent working atmosphere and fruitful discussions during the course of this work.

7. REFERENCES

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Table 1. Experimental results of successive rotations

Inter- polator	Parameters		SNR			
	ω	γ_{10}	Baboon1	Geom	Barbara	Baboon2
o-moms			25.67	21.57	24.54	24.38
β^3			24.45	19.76	21.72	22.73
$\beta^{31}, \alpha=0.6$	2	-0.062	26.13	22.19	25.75	24.98
$\beta^{31}, \alpha=0.6$	0.5	-0,0686	26.33	22.41	26.28	25.23
$\beta^{31}, \alpha=0.6$	1	-0,0748	26.36	22.40	26.30	25.26
$\beta^{31}, \alpha=0.7$	0.2	-0,0714	26.37	22.44	26.36	25.27
$\beta^{31}, \alpha=0.7$	0.5	-0,814	26.09	21.98	25.54	24.90
$\beta^{31}, \alpha=0.8$	0.2	-0,098	25,74	21.30	24.76	24.47

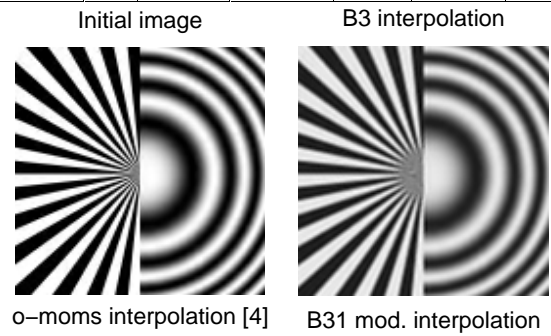


Figure 2. Central part of the Geometrical image
Initial image B3 interpolation

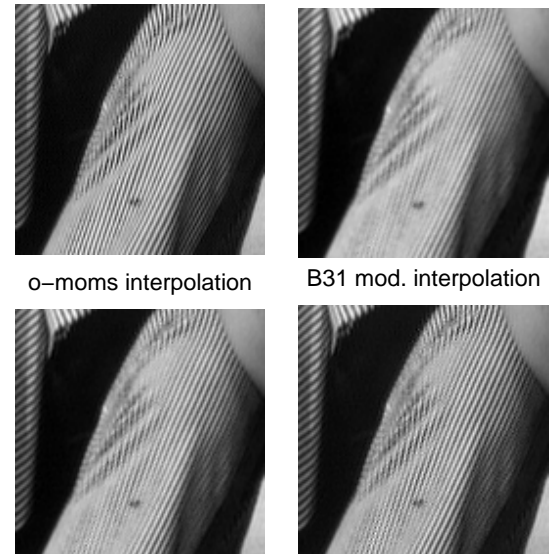


Figure 3. Right lower corner of the Barbara image
o-moms interpolation B31 mod. interpolation