EFFICIENT DESIGN OF FIR DIGITAL DIFFERENTIATORS USING LEAST-SQUARES APPROACH

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ABSTRACT

This paper presents new simple and compact formulas for design of first-order digital differentiators based on the least-squares method. Two different cases, fullband and non-fullband differentiators, are considered. Using the proposed technique we avoid solving the system of linear equations for the case of fullband differentiators. A great number of design examples and plots are given.

1. INTRODUCTION

Recently, many techniques exist for designing digital differentiators (DD). The minimax approach [1-4] is one of the oldest and frequently used methods. Other methods are based on the Fourier series in conjunction with the Kaiser window function [5,6] or special accuracy constraints [12]. The 'eigenfilter' method [7,8] can also be applied for first- and higher-order DD. This approach is based on the computation of an eigenvector of a real, symmetric and positive-definite matrix. The main advantage over the McClellan-Parks algorithm [2] is that it is easy to incorporate time and frequency domain constraints with comparable design time.

Least squares technique (LS) for design of DD [9] is a good alternative to the minimax method. There are a lot of modifications of this method with different properties (accuracy, computational complexity, and theoretical basis). For example, an iterative procedure for design of equiripple DD based on the weighted LS approach is given in [10,11]. To reduce a computation time a prediction technique for the length of DD is applied.

Efficient design of differentiators with maximal linearity for midband frequency ranges (i.e. $\omega=\pi/2$) is shown in [12,13]. In addition, mathematical relations for the weighting coefficients are successfully derived. A similar method [14,15] is developed for maximally flat DD for the low frequencies region. Other useful approaches are given in [16-18].

This paper proposes new compact formulas for least-squares design of differentiators. Using this analytical solution, we can avoid solving a system of linear equations for the case of fullband differentiators. Some illustrative examples and comparison with McClellan-Parks algorithm are shown.

2. THEORETICAL BACKGROUND FOR DESIGN OF DIFFERENTIATORS

For the design of linear-phase DD the impulse response $h(n)$ is required to be antisymmetrical, i.e.: $h(n)=-h(N-1-n)$, with $h((N-1)/2)=0$ when N is odd. The frequency response could be expressed as: $H(e^{j\omega}) = M(\omega)e^{j(\pi/2-\omega(N-1)/2)}$, where:

$$M(\omega) = \begin{cases} \sum_{n=1}^{(N-1)/2} b(n).\sin n\omega \ , & N \text{ odd} \\ \sum_{n=1}^{N/2} b(n).\sin(n-1/2)\omega \ , & N \text{ even} \ . \end{cases} \tag{1}$$

If N is odd, $b(n)=2h[(N-1)/2-n]$ for $1\le n \le (N-1)/2$. If N is even, $b(n)=2h(N/2-n)$ for $1\le n \le N/2$.

An ideal k-th order differentiator has a frequency response $H_I(e^{j\omega}) = D(\omega)e^{jk\pi/2}$. For the case of first-order DD with k=1:

$$H_I(e^{j\omega}) = D(\omega)e^{j\pi/2}, \tag{2}$$

where $D(\omega)=\omega$ for $0 \le \omega \le \omega_p \le \pi$, and ω_p is the passband edge frequency. In other words, ω_p is the highest frequency for which differentiating action is required.

On comparing equations for $H(e^{j\omega})$ and $H_I(e^{j\omega})$, it can be seen that a nonrecursive DD with linear phase can be designed if $M(\omega)$ approximates $D(\omega)$. A fullband DD (with $\omega_p=\pi$) can be designed only when N is even, since, when N is odd, $M(\omega)$ becomes zero when $\omega=\pi$.

3. LEAST-SQUARES APPROACH

3.1. Error function minimization

We consider the least-square difference between desired amplitude response $D(\omega)$ and the actual one $M(\omega)$ over the passband of differentiator:

$$E_{LS} = \frac{1}{\pi}\int_0^{\omega_p}\left[D(\omega) - M(\omega)\right]^2 d\omega\,, \tag{3}$$

where $\omega_p = 2\pi.f_p$.

Now, let us define:

$$\mathbf{b} = \begin{cases} \left[b(1), b(2),...., b((N-1)/2)\right]^T, & N \text{ odd} \\ \left[b(1), b(2),...., b(N/2)\right]^T, & N \text{ even} \end{cases} \tag{4}$$

and

$$\mathbf{c}(\omega) = \begin{cases} \left[\sin\omega, \sin 2\omega,...., \sin\left(\frac{N-1}{2}\omega\right)\right]^T, & N \text{ odd} \\ \left[\sin\frac{1}{2}\omega, \sin\frac{3}{2}\omega,...., \sin\left(\frac{N-1}{2}\omega\right)\right]^T, & N \text{ even}\,, \end{cases} \tag{5}$$

where superscript T is the vector transpose operation. Then $M(\omega)$ can be written as:

$$M(\omega)= \mathbf{b}^T \mathbf{c}(\omega) . \quad (6)$$

By minimizing the error function E_{LS} with respect to the filter coefficients, the required filter is designed. As a result, we obtain a system of linear equations **Qb=d**, where

$$\mathbf{Q} = \int_0^{\omega_p} \mathbf{c}(\omega)\mathbf{c}^T(\omega)d\omega \ , \quad \mathbf{d} = \int_0^{\omega_p} D(\omega)\mathbf{c}(\omega)d\omega. \quad (7)$$

Q is a positive-defined, real, and symmetric matrix.

3.2. Compact formulas for the entries of d and Q

After evaluation the integrals from (7) we can find the following new compact formulas for the entries of **d** and **Q** [19]:

$$d(n) = \begin{cases} \dfrac{\omega_p}{n}\left[\operatorname{sinc} n\omega_p - \cos n\omega_p\right], & N \text{ odd}, \quad 1 \le n \le \dfrac{N-1}{2} \\ \dfrac{\omega_p}{n-1/2}\left[\operatorname{sinc}(n-1/2)\omega_p - \cos(n-1/2)\omega_p\right], & \\ & N \text{ even}, \quad 1 \le n \le \dfrac{N}{2} \end{cases} \quad (8)$$

and

$$q(n,m) = \begin{cases} \dfrac{\omega_p}{2}\left[\operatorname{sinc}(n-m)\omega_p - \operatorname{sinc}(n+m-l)\omega_p\right], & n \ne m \\ \dfrac{\omega_p}{2}\left[1 - \operatorname{sinc}(2n-l)\omega_p\right], & n = m \\ \text{and} \quad l = 0 \quad \text{for} \quad 1 \le n,m \le \dfrac{N-1}{2}, & N \text{ odd} \\ l = 1 \quad \text{for} \quad 1 \le n,m \le \dfrac{N}{2}, & N \text{ even} \end{cases} \quad (9)$$

where sinc x = sinx/x.

4. DESIGN OF FULLBAND DD

For the case of fullband differentiators ($\omega_p=\pi$, only when N is even) we obtain very simple relations for the entries of **d** and **Q**:

$$d(n) = \frac{4.(-1)^{n+1}}{(2n-1)^2}, \quad 1 \le n \le \frac{N}{2} \quad (10)$$

and

$$q(n,m) = \begin{cases} \dfrac{\pi}{2}, & n = m \\ 0, & n \ne m \end{cases} \quad 1 \le n,m \le \frac{N}{2}. \quad (11)$$

By that means, we avoid solving the system of linear equations and directly find:

$$b(n) = \frac{8.(-1)^{n+1}}{\pi.(2n-1)^2}, \quad 1 \le n \le \frac{N}{2}. \quad (12)$$

Therefore, the amplitude response of fullband DD can be described as:

$$M(\omega) = \frac{8}{\pi}\sum_{n=1}^{N/2} \frac{(-1)^{n+1}}{(2n-1)^2} \sin(n-1/2)\omega \ , \quad N \text{ even}. \quad (13)$$

Using (12), the impulse response is expressed as follow:

$$h\left(\frac{N}{2}-n\right) = \frac{4.(-1)^{n+1}}{\pi.(2n-1)^2} \ , \quad h\left(\frac{N}{2}-n\right) = -h\left(\frac{N}{2}-1+n\right),$$

for $1\le n \le N/2$. Consequently, the transfer function $H(z)$ of fullband DD designed by least-squares method is:

$$H(z) = \sum_{n=1}^{N/2} \frac{4.(-1)^{n+1}}{\pi.(2n-1)^2}\left(z^{-v} - z^{-(v+s)}\right),$$

where $v=N/2-n$ and $s=1,3,5, ... ,N-1$.

5. DESIGN EXAMPLES AND CONCLUSIONS

We develop a MatLab based program to demonstrate the flexibility and effectiveness of the new relations. Different illustrative plots for fullband and non-fulband DD are given in Fig.1–Fig.6. Amplitude responses obtained by our approach are shown in Fig.1, Fig.3, and Fig.4. The error function $E=D(\omega)-M(\omega)$ against the frequency is given in Fig.4–Fig.6. Peak values of the amplitude responses against the length N of DD achieved for three different values of ω_p are plotted in Fig.2. A comparison between the proposed approach and the minimax method is given in Fig.6. We obtain better error function for fullband DD with the new formulas in most of the frequency band, except in the narrow-band region near the cutoff edge.

The error function for non-fullband case is close to this one of minimax method. As a result, very low errors are achieved (for example, we have a peak error $0.703.10^{-7}$ or -143dB for DD with $N=41$, $\omega_p=0.74\pi$).

In conclusion, we could say that the new expressions allow a very fast and precise design of fullband and non-fullband differentiators. Graphical results and comparison present the proposed approach as a good alternative of minimax and eigenfilter methods.

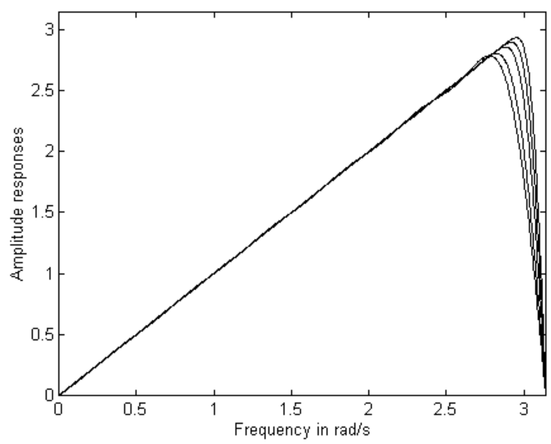

Figure 1. Five amplitude responses of non-fullband DD with $\omega_p=0.9\pi$ for different values of N (from left to right: $N=25, 33, 55, 81, 119$).

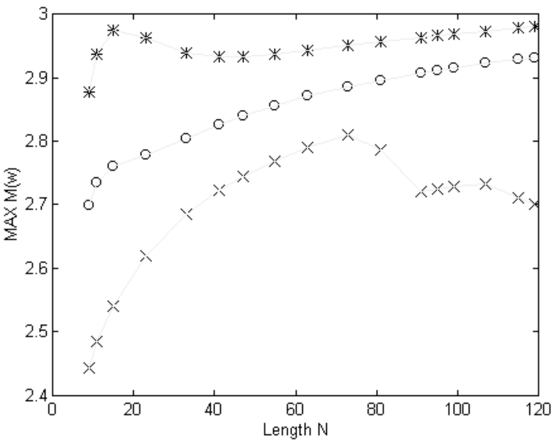

Figure 2. Peak values of the amplitude responses against length N obtained for three different passband edge frequencies (x – ω_p=0.83π, o – ω_p=0.9π, ∗ – ω_p=0.94π).

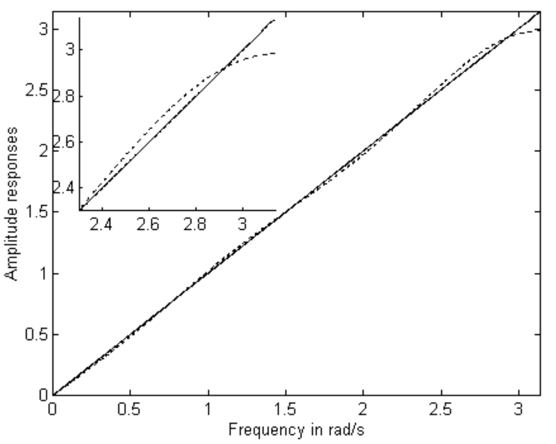

Figure 3. Amplitude responses of fullband DD: the solid line corresponds to N=116, the dotted line – to N=8.

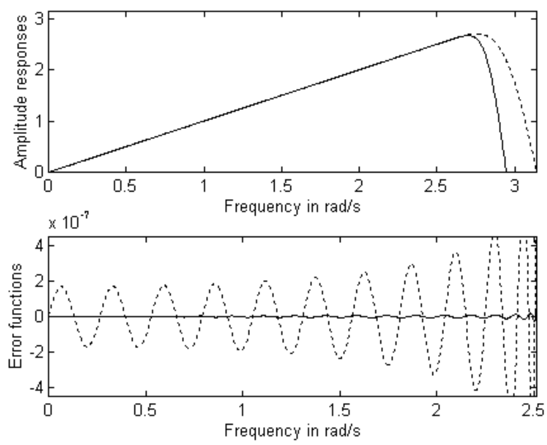

Figure 4. Amplitude and error curves for non-fullband DD with ω_p=0.8π, N=73 (solid line) and ω_p=0.8π, N=45 (dotted line).

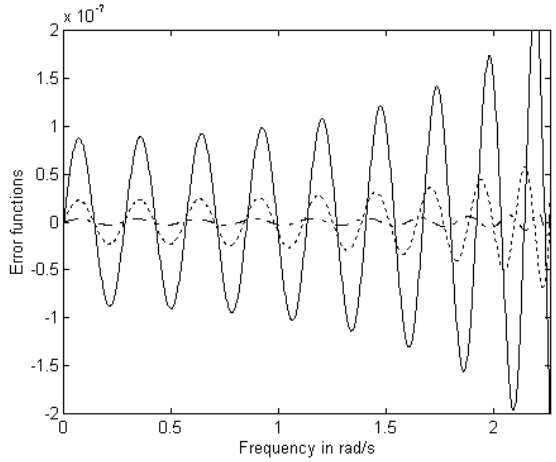

Figure 5. Error functions against the frequency for non-fullband DD with N=41 and ω_p=0.77π, 0.75π, and 0.72π (indicated by solid, dotted and dashdot lines, respectively).

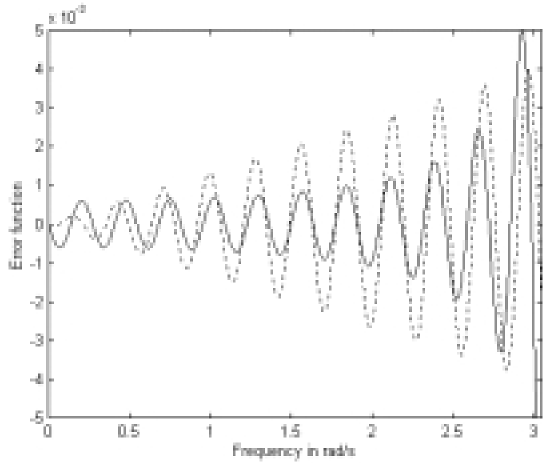

Figure 6. The error curves for fullband differentiator with length N=46 (solid line: new LS approach, dotted line: McClellan-Parks algorithm).

6. REFERENCES

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7. A MATLAB PROGRAM

```
% Efficient design of FIR digital
% differentiators using least-squares
% approach
% Author: Guergana S. Mollova
% Dept.of CAD, UACG-Sofia, 1999
% N-length of DD
% wp-passband edge frequency

clear
%Input data
N=45;        %Example data N=45, wp=0.8pi
wp=0.8*pi;

%Elements of vector d
if rem(N,2)==0    %N - even
  for n=1:N/2
   d(n)=sin((n-1/2)*wp)/((n-1/2)^2)-cos((n-
        1/2)*wp)*wp/(n-1/2);
  end;
 else             %N - odd
  for n=1:(N-1)/2
   d(n)=sin(n*wp)/(n^2)-(wp*cos(n*wp))/n;
  end; %for
end; %if

%Elements of matrix Q
if rem(N,2)==0    %N - even
  for n=1:N/2
   for m=1:N/2
    if n~=m
     q(n,m)=sin((n-m)*wp)/(2*(n-m))-
            sin((n+m-1)*wp)/(2*(n+m-1));
    else
     q(n,m)=-sin(2*(n-1/2)*wp)/(4*(n-
              1/2))+wp/2;
    end; %if
   end; %for
  end; %for
else              %N - odd
  for n=1:(N-1)/2
   for m=1:(N-1)/2
    if n~=m
     q(n,m)=sin((n-m)*wp)/(2*(n-m))-
            sin((n+m)*wp)/(2*(n+m));
    else
     q(n,m)=-sin(2*n*wp)/(4*n)+wp/2;
    end; %if
   end; %for
  end; %for
end; %if

% System of linear equations
b=q\d'; % Gaussian elimination

b1=b';   %b-transpose
```