A THREE DIMENSIONAL WINDOW FUNCTION FOR THE CONSISTENT ESTIMATION OF THE TRISPECTRUM

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ABSTRACT

Window functions are used in the estimation of power spectra, [1] and bispectra [2] in order to ensure the consistency of the periodogram and the Fourier type bispectrum estimation methods. In this paper, a three dimensional optimum bias lag window function is introduced in the estimation of the fourth order cumulant spectrum, also called trispectrum, which is estimated from the three dimesional Fourier transform of the fourth order cumulants. The bias of the window function is also developed in the paper.

1. INTRODUCTION

Although spectral theory is a well established approach to many signal processing tasks, its applicability depends heavily upon the assumption of the linearity and Gaussianity of the signal and/or minimum phase assumption of the underlying model. However, higher order statistics, which has been used widely in recent years for signal processing tasks [3, 4] assumes that the underlying signal is non-Gaussian and it can identify nonminimum phase signals. Moreover, higher order spectra can reveal the phase information unlike the power spectrum, and are zero for the additive Gaussian noise. Additionally, the importance of the higher order spectra in the analysis of nonlinear systems are discussed in [4, 5] Most of the algorithms in the literature are developed for the third order cumulants and their spectrum also called bispectrum. The trispectrum, the Fourier transform of the fourth order cumulants, is zero only for Gaussian random processes while the bispectrum is zero for any symmetrically distributed random processes. Therefore, for applications where the underlying signal could be symmetrically distributed, the trispectrum should be used as a higher order statistical tool. Moreover, the bispectrum and trispectrum together form a more powerful instrument in the analysis of nonlinear time series models than either one when used alone.

Brillinger and Rosenblatt [6] developed the asymptotic theory of the k-th order spectra, including the asymptotic theory of the trispectrum. The trispectrum is an inconsistent estimate as is the power spectrum and the bispectrum when estimated through nonparametric methods. Brillinger and Rosenblatt [6] and Rosenblatt [7] presented consistent estimators for the k-th order spectra when the smoothing is done in the frequency domain. A specific three dimensional window function itself is not developed in these references. To our knowledge, such a window function has not been developed in the literature.

2. THREE DIMENSIONAL OPTIMUM LAG WINDOW

The trispectrum is

$$C(\omega_1,\omega_2,\omega_3) = \sum_{m_1=-L}^{L}\sum_{m_2=-L}^{L}\sum_{m_3=-L}^{L} w(m_1,m_2,m_3) \\ c(m_1,m_2,m_3)exp(-j(\omega_1 m_1,\omega_2 m_2,\omega_3 m_3) \quad (1)$$

where $w(m_1,m_2,m_3)$ and $c(m_1,m_2,m_3)$ are the window function and the fourth order cumulant respectively.

The window function $w(m_1,m_2,m_3)$ should satisfy the following constraints:

1. By taking into account the symmetries of the fourth order cumulants $c(m_1,m_2,m_3)$ one may assume that $w(m_1,m_2,m_3)$ has the same symmetries as the fourth order cumulants.
2. By restricting the region of the time lag, it is assumed that

   $$w(m,m,m) = 0, \quad (2)$$

   $$|m| = max(|m_1|,|m_2|,|m_3|,|m_1-m_2|, \\ |m_2-m_3|,|m_1-m_3|) > L,$$

   where L is the maximum lag of the window.
3. The window is normilized such that,

   $$w(0,0,0) = 1 \quad (3)$$

4. Nonnegativeness of the window is defined with the condition,

   $$W(\omega_1,\omega_2,\omega_3) \geq 0 \;\; for\ all \;\; (\omega_1,\omega_2,\omega_3) \quad (4)$$

   where $W(\omega_1,\omega_2,\omega_3)$ is the Fourier transform ofthe window.

   Here as a class of functions which satisfy the conditions mentioned above with the symmetry relation and equations 2, 3 and 4 , we express the three dimensional optimum lag window $w(m_1,m_2,m_3)$ as follows

$$w(m_1, m_2, m_3) = d(m_1)d(m_2)d(m_3)$$
$$d(m_3 - m_2)d(m_3 - m_1)d(m_2 - m_1) \qquad (5)$$

where an even function $d(m)$ satisfies the following properties

$$\begin{aligned} d(m) &= d(-m) \\ d(m) &= 0 \;\; for \;\; m > L \\ d(0) &= 1 \end{aligned} \qquad (6)$$

$$D(\omega) \geq 0, \;\; for \;\; all \;\; \omega \qquad (7)$$

Equations 5, 6, and 7 allow the reconstruction of three dimensional window functions for trispectrum estimation using standard one-dimensional lag windows. The function $d(.)$ could be any one dimensional lag window as long as it satisfies the constraints in equations 6, 7. However, not all conventional, one-dimensional windows satisfy constraint 7. Windows that satisfy equations 6, 7 are given in [2].

Here $d(.)$ is chosen to be optimum window for one dimension given also in [8],

$$d(m) = \begin{cases} \frac{1}{\pi}|\sin \pi \frac{m}{L}| + (1 - \frac{|m|}{L})(\cos \frac{\pi m}{L}), & \text{if } |m| \leq L \\ 0 & |m| > L \end{cases} \qquad (8)$$

The function $d(.)$ in eqn. 8 is the optimum window because it is the minimum bias window in one dimension [8]. It is shown in Appendix that the three-dimensional optimum lag window function given in eqn. 8, together with eqn. 5, satisfies the given constraints (the symmetries and constraints in 2, 3, and 4.

3. VARIANCE

The asymptotic variance of the trispectrum is derived by Brillinger [9]. It is as follows:

$$\sigma^2 = \frac{V\;K^2}{M} H(\omega_1)H(\omega_2)H(\omega_3)H^*(\omega_1 + \omega_2 + \omega_3) \qquad (9)$$

where $*$ is the conjugate operation, K is the number of frames of size M, N is the sample size, and V is the energy of the window , which is defined as:

$$V \triangleq \sum_{m=-L}^{L} \sum_{n=-L}^{L} \sum_{l=-L}^{L} |\omega(m, n, l)|^2. \qquad (10)$$

4. BIAS

Here, we will discuss the bias of the trispectrum and the way of obtaining the optimum lag window in terms of bias.

The trispectrum estimate is convolved with a suitable window $W(\omega_1, \omega_2, \omega_3)$ to give the required smoothed trispectrum

$$\hat{C}(\omega_1, \omega_2, \omega_3) =$$
$$(\frac{1}{2\pi})^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(\omega_1 - y_1, \omega_2 - y_2, \omega_3 - y_3)$$
$$W(y_1, y_2, y_3)dy_1 dy_2 dy_3. \qquad (11)$$

The trispectrum can be approximated by the true trispectrum $C(\omega_1, \omega_2, \omega_3)$ for sufficiently large data. The bias of the trispectrum is

$$E\{C(\omega_1, \omega_2, \omega_3)\} =$$
$$\frac{1}{2\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(\omega_1 - y_1, \omega_2 - y_2, \omega_3 - y_3)$$
$$W(y_1, y_2, y_3)dy_1 dy_2 dy_3 \qquad (12)$$

If the trispectrum of the process is assumed to have continuous second derivatives, it follows from the mean value theorem [10] that

$$C(\omega_1 - y_1, \omega_2 - y_2, \omega_3 - y_3) =$$
$$C(\omega_1, \omega_2, \omega_3) -$$
$$\{y_1 C'_1(\omega_1, \omega_2, \omega_3) + y_2 C'_2(\omega_1, \omega_2, \omega_3) +$$
$$y_3 C'_3(\omega_1, \omega_2, \omega_3)\}$$
$$\frac{1}{2}\{y_1^2 C''_{11}(u_1, u_2, u_3) + 2y_1 y_2 C''_{12}(u_1, u_2, u_3) +$$
$$2y_1 y_3 C''_{13}(u_1, u_2, u_3) + 2y_2 y_3 C''_{23}(u_1, u_2, u_3) +$$
$$y_2^2 C''_{22}(u_1, u_2, u_3) + y_3^2 C''_{33}(u_1, u_2, u_3)\} \qquad (13)$$

where $u_1 = \omega_1 - \Theta_1(y_1)y_1, \;\; u_2 = \omega_2 - \Theta_2(y_2)y_2, \;\; u_3 = \omega_3 - \Theta_3(y_3)y_3$ and $|\Theta(y_1)| \leq 1 \quad (i = 1, 2, 3)$

$C'_i(\omega_1, \omega_2, \omega_3)$ indicates the first derivative of $C(\omega_1, \omega_2, \omega_3)$ with respect to ith argument, and $C''_{ij}(u_1, u_2, u_3)$ indicates the second derivative with respect to ith and jth arguments.

From eqn. 2, 4 and 13, the bias $\Delta C(\omega_1, \omega_2, \omega_3)$ can easily be written in the form

$$\Delta C(\omega_1, \omega_2, \omega_3) =$$
$$\frac{1}{2(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{y_1^2 C''_{11}(u_1, u_2, u_3) +$$
$$2y_1 y_2 C''_{12}(u_1, u_2, u_3) + 2y_1 y_3 C''_{13}(u_1, u_2, u_3) +$$
$$2y_2 y_3 C''_{23}(u_1, u_2, u_3) + y_2^2 C''_{22}(u_1, u_2, u_3) +$$
$$y_3^2 C''_{33}(u_1, u_2, u_3)\}|_{\substack{u_1 = \omega_1 - \Theta_1(y_1)y_1 \\ u_2 = \omega_2 - \Theta_2(y_2)y_2, \quad u_3 = \omega_3 - \Theta_3(y_3)y_3}}$$
$$W(y_1, y_2, y_3)dy_1 dy_2 dy_3 \;\; . \qquad (14)$$

For large T and an acceptable variance, $W(\omega_1, \omega_2, \omega_3)$ should take significant values only in a sufficiently small interval $(-\epsilon_1, \epsilon_1) * (-\epsilon_2, \epsilon_2) * (-\epsilon_3, \epsilon_3)$. If the second derivatives of $C(\omega_1, \omega_2, \omega_3)$ do not vary significantly in the area $(\omega_1 - \epsilon_1, \omega_1 + \epsilon_1) * (\omega_2 - \epsilon_2, \omega_2 + \epsilon_2) * (\omega_3 - \epsilon_3, \omega_3 + \epsilon_3)$, eqn. 14 can be approximated as follows:

$$\Delta C(\omega_1,\omega_2,\omega_3) \simeq \frac{C''(\omega_1,\omega_2,\omega_3)}{2(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y_1^2+y_2^2+y_3^3) W(\omega_1,\omega_2,\omega_3)dy_1dy_2dy_3 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2y_1y_2W(\omega_1,\omega_2,\omega_3)dy_1dy_2dy_3 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2y_2y_3W(\omega_1,\omega_2,\omega_3)dy_1dy_2dy_3 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2y_1y_3W(\omega_1,\omega_2,\omega_3)dy_1dy_2dy_3 \quad (15)$$

Hence if $|C''(\omega_1,\omega_2,\omega_3)| \leq D$
where $D = C_m = \max_{i,j=1,2}[|C''_{ij}(\omega_1,\omega_2,\omega_3)|]$, the bias cannot exceed

$$\triangle|C(\omega_1-y_1,\omega_2-y_2,\omega_3-y_3)| \leq \frac{C_m}{2(2\pi)^3}\int\int\int(y_1^2+y_2^2+y_3^3)W(\omega_1,\omega_2,\omega_3)dy_1dy_2dy_3 + |\int\int\int 2y_1y_2W(\omega_1,\omega_2,\omega_3)dy_1dy_2dy_3 + \int\int\int 2y_2y_3W(\omega_1,\omega_2,\omega_3)dy_1dy_2dy_3 + \int\int\int 2y_1y_3W(\omega_1,\omega_2,\omega_3)dy_1dy_2dy_3| \quad (16)$$

Therefore, the problem reduces to that of seeking the window that minimises the following integral, subject to constraints 2-4:

$$J = \frac{1}{(2\pi)^3}\int\int\int(y_1-y_2-y-3)^2 W(y_1,y_2,y-3)dy_1dy_2dy_3 \quad (17)$$

This is the specification for the optimum trispectral window and in this case the bias of the estimate of $C(\omega_1,\omega_2,\omega_3)$ does not exceed $C_m J_{min}/2$.

In fact, the right side of the equation 17 can be written by using the properties of the Fourier transform as:

$$\frac{\partial^2\omega(m,m,m)}{\partial^2 m}|_{m=0} = \frac{-1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y_1+y_2+y_3)^2 W(y_1,y_2,y_3)dy_1dy_2dy_3\ . \quad (18)$$

We would like to make the connection with eqn. 5 here. Equation 5 is a class of functions which satisfies the condition given by eqn. 6. By taking the second derivative of $w(m,m,m)$ from eqn. 5 at $m=0$,

$$J = -3d''(0) = \frac{3}{2\pi}\int_{-\infty}^{\infty}\omega^2 D(\omega)d\omega. \quad (19)$$

Thus, the problem is reduced to finding the one-dimensional window which minimises eqn. 19 under constraints 6 and 7. Appropriate functions are given in [8] which is eqn. 8.

5. CONCLUSION

In this paper, we have introduced the three dimensional window function to ensure consistency of the tripspectrum. The bias of the tripspectrum is also developed. It is proved that optimum bias window function for one dimension also satisies optimum bias for the three dimension window when replaced in the function.

6. APPENDIX

Here, we will show the three-dimensional optimum lag window function given in eqn. 8, together with eqn. 5, satisfies the given constraints (the symmetries and constraints in 2, 3, and 4.

When we substitute $d(m)$ in eqn. 8 into eqn.5 it clearly satisfies eqn.2 and eqn.3 . We briefly mention about the symmetries of the fourth order cumulants before we show that the window function introduced also satisfies the symmetry properties of the fourth order cumulants.

The fourth order cumulant of a zero mean, stationary random process $x(n)$ is

$$c_4(m_1,m_2,m_3) = E\{x(n)x(n+m_1)x(n+m_2)(n+m_3)\} -[r(m_1)r(m_2-m_3)+r(m_2)r(m_3-m_1)+ r(m_3)r(m_1-m_2)] \quad (20)$$

where r is the autocorrelation function and E is the expectation operator.

Similar symmetry properties are valid for the fourth order cumulants as in third order cumulants but when defining these symmetries the (2nd order cumulant) autocorrelation terms in the calculation of the fourth order cumulants should be also considered. For example for $c(m,n,l) = c(m-l,n-l,-l)$

$$c(m-l,n-l,-l) = \frac{1}{N}\sum_{s_1}^{s_2} y(i)y(i+(m-l))y(i+(n-l))y(i+(-l)) -r(m-l)r(n-l+l)-r(n-l)r(-l-m+l) -r(-l)r(m-l-n+l) \quad (21)$$

where,

$$s_1 = i = max(0,-(m-l),-(n-l),-(-l))$$
$$s_2 = min(N, N-(m-l), N-(n-l), N-(-l)) \quad (22)$$

using the symmetries of autocorrelation, the autocorrelation terms become,

$$-r(n-l)r(m) - r(l-m)r(n) - r(l)r(m-n) \tag{23}$$

There are 24 symmetry regions for the fourth order cumulants. They are not going to be given here because of the space limitation. In order to show that window function also satisfies the symmetries of the fourth order cumulants, only one of the symmetry condition will be shown, the rest are trivial.

$$\omega(m_1, m_2, m_3) = \omega(-m_1, m_2 - m_1, m_3 - m_1) \tag{24}$$

By using eqn.5 and eqn.8

$$\begin{aligned} &\omega(m_1, m_2, m_3) = \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_1}{L}| + (1 - \frac{|m_1|}{L})(\cos\frac{\pi m_1}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_2}{L}| + (1 - \frac{|m_2|}{L})(\cos\frac{\pi m_2}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_3}{L}| + (1 - \frac{|m_3|}{L})(\cos\frac{\pi m_3}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_3 - m_2}{L}| + (1 - \frac{|m_3 - m_2|}{L})(\cos\frac{\pi(m_3 - m_2)}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_3 - m_1}{L}| + (1 - \frac{|m_3 - m_1|}{L})(\cos\frac{\pi(m_3 - m_1)}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_2 - m_1}{L}| + (1 - \frac{|m_2 - m_1|}{L})(\cos\frac{\pi(m_2 - m_1)}{L})] \end{aligned} \tag{25}$$

$$\begin{aligned} &\omega(-m_1, m_2 - m_1, m_3 - m_1) = \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{-m_1}{L}| + (1 - \frac{|-m_1|}{L})(\cos\frac{\pi(-m_1)}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_2 - m_1}{L}| + (1 - \frac{|m_2 - m_1|}{L})(\cos\frac{\pi(m_2 - m_1)}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_3 - m_1}{L}| + (1 - \frac{|m_3 - m_1|}{L})(\cos\frac{\pi(m_3 - m_1)}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_3 - m_2}{L}| + (1 - \frac{|m_3 - m_2|}{L})(\cos\frac{\pi(m_3 - m_2)}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_3}{L}| + (1 - \frac{|m_3|}{L})(\cos\frac{\pi(m_3)}{L})] \\ &\quad [\frac{1}{\pi}|\sin\pi\frac{m_2}{L}| + (1 - \frac{|m_2|}{L})(\cos\frac{\pi(m_2)}{L})] \end{aligned} \tag{26}$$

Because the cosine function is an even function, eqn.25 and eqn.26 are equal.

7. REFERENCES

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