THE FRACTIONAL FOURIER TRANSFORM: A TUTORIAL

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ABSTRACT

The fractional Fourier transfrom is an important tool for both signal processing and optical communities. This paper presents a tutorial which includes the major related aspects of this transformation.

1. INTRODUCTION

The fractional Fourier transform (FRT) operation was shown to be useful for various spatial filtering and signal processing applications [1]-[8]. The FRT is a private case of the $ABCD$ matrix. When the $ABCD$ matrix accept the form of:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \qquad (1)$$

the $ABCD$ transform becomes the FRT.

In this transform the amount of shift variance may be controlled by choosing the proper fractional order p for the transformation while ϕ is $\phi = \frac{p\pi}{2}$. When the fractional order is one, the FRT becomes the conventional Fourier transform which is totally shift invariant. For fractional order of zero the FRT gives the input function, i.e totally shift variant. For any other fractional orders in between, the transform has a partial amount of shift variance.

2. FRT- DEFINITION

There are two common interpretations for the FRT. Both definitions were proven to be identical as shown in Ref. [7].

2.1. Definition based on propagation in graded index media

The first FRT definition [9, 10, 11] is based on the field propagating along a quadratic graded index (GRIN) medium having a length proportional to p (p being the FRT order). The eigen-modes of quadratic GRIN media are the Hermite-Gaussian (HG) functions, which form an orthogonal and complete basis set. The mth member of this set is expressed as

$$\Psi_m(x) = H_m\left(\frac{\sqrt{2}x}{\omega}\right)\exp\left(-\frac{x^2}{\omega^2}\right), \qquad (2)$$

where H_m is a Hermite polynomial of order m and ω is a constant associated with the GRIN medium parameters. An extension to two lateral coordinates x and y is straightforward, with $\Psi_m(x)\Psi_n(y)$ as elementary functions.

The propagation constant for each HG mode is given by:

$$\beta_m = k\sqrt{1-\frac{2}{k}\sqrt{\frac{n_2}{n_1}}(m+\frac{1}{2})} \approx k-\sqrt{\frac{n_2}{n_1}}(m+\frac{1}{2}). \quad (3)$$

with $k = 2\pi/\lambda$. The HG set is used to decompose any arbitrary distribution $u(x)$

$$u(x) = \sum_m A_m\Psi_m(x), \qquad (4)$$

where the coefficient A_m of each mode $\Psi_m(x)$ is given by:

$$A_m = \int_{-\infty}^{\infty} u(x)\Psi_m(x)/h_m\, dx, \qquad (5)$$

with $h_m = 2^m m!\sqrt{\pi}\omega/\sqrt{2}$.

Using the above decomposition, the FRT of order p is defined as

$$\mathcal{F}^p[u](x) = \sum_m A_m\Psi_m(x)\exp(i\beta_m pL) \qquad (6)$$

$L = (\pi/2)\sqrt{n_1/n_2}$ is the GRIN length that results in the conventional Fourier transform. It was shown [10] that this definition agrees well with the classical Fourier transform definition when $p = 1$.

2.2. Definition based on Wigner distribution function

A complete signal description, displaying space and frequency information simultaneously, can be achieved by the space-frequency WDF [12].

In Ref. [13] the fractional Fourier transform operation is defined by following the signal $u(x)$ while its WDF is rotated by an angle $\phi = p\pi/2$. Note that the WDF of a 1-D function is a 2-D function and the rotation interpretation is easily displayed. In Ref. [13], the same rotation strategy was generalized to 2 D signals, i.e. images, whose WDFs are 4-D distributions. The WDF of a function can be rotated with bulk optics. It was suggested [13] to use the optical system of Fig. 1 for implementing the FRT operator.

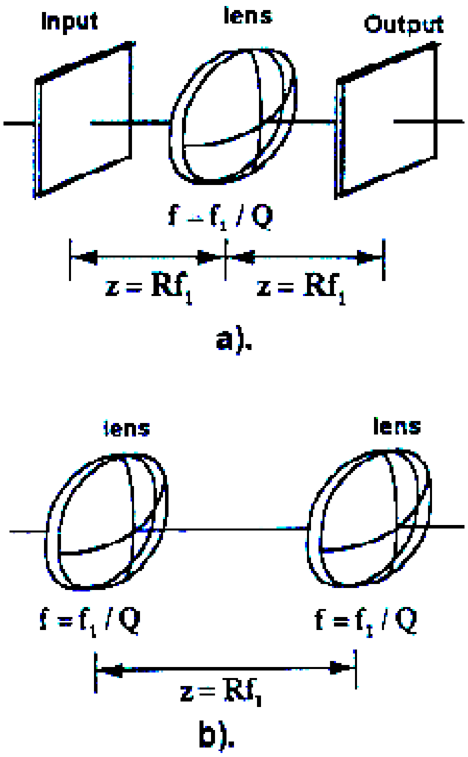

Figure 1: The two possible optical setups for obtaining the FRT. a). Type I configuration. b). type II configuration.

This optical setup represents in the WDF space three shearing operations: x, ν, x shearing or ν, x, ν-shearing. Where ν is the spectral coordinate and x is the spatial one. The x-shearing is performed by free-space propagation , then a lens performs ν-shearing, then again x-shearing is performed by free-space propagation. In his paper [13], Lohmann characterized this optical system using two parameters, Q and R

$$f = f_1/Q \qquad , \qquad z = f_1 R, \qquad (7)$$

where f_1 is an arbitrary length, f is the focal length of the lens and z is the distance between the lens and the input (or output) plane. As known from Ref. [13] for an FRT of order p, Q and R should be chosen as:

$$R = \tan(\phi/2) \qquad , \qquad Q = \sin(\phi) \qquad (8)$$

for the type I configuration and as

$$R = \sin(\phi) \qquad , \qquad Q = \tan(\phi/2) \qquad (9)$$

for the type II configuration. Note that $\phi = p(\pi/2)$.

By analyzing the optical configuration of Fig. 1, Lohmann (Ref [13]) obtained:

$$\begin{aligned} u_p(x) &= \mathcal{F}^p[u(x_0)] \\ &= C_1 \int_{-\infty}^{\infty} u(x_0) \exp\left(i\pi \frac{x_0^2 + x^2}{\lambda f_1 \tan\phi}\right) \cdot \\ &\quad \cdot \exp(-i2\pi \frac{x x_0}{\lambda f_1 \sin\phi}) dx_0 \qquad (10) \end{aligned}$$

with

$$C_1 = \frac{\exp\left[-i\left(\frac{\pi \mathrm{sgn}(\sin\phi)}{4} - \frac{\phi}{2}\right)\right]}{|\lambda f_1 \sin\phi|^{\frac{1}{2}}} \qquad (11)$$

This last equation defines the FRT for 1-D functions with λ as a wavelength. Generalization for 2-D functions is straightforward. Note that λf_1 is also coined the scaling factor.

The two interpretations of the FRT operation have been united into one formulation through a transformation kernel, as illustrated in Ref. [8]:

$$u_p(x) = \{\mathcal{F}^p[u(x')]\}(x) = \int_{-\infty}^{\infty} B_p(x, x') u(x') dx' \qquad (12)$$

where $B_p(x, x')$ is the kernel of the transformation and p is the fractional order. The kernel has two optical interpretations, one as a propagation through GRIN medium [10]

$$\begin{aligned} B_p(x, x') &= \sqrt{2} \exp\left[-\frac{1}{w}(x^2 + x'^2)\right] \sum_{n=0}^{\infty} \frac{i^{-pn}}{2^n n!} \cdot \\ &\quad \cdot H_n\left(\frac{\sqrt{2}}{w} x\right) H_n\left(\frac{\sqrt{2}}{w} x'\right) \qquad (13) \end{aligned}$$

and the second as a rotation operation applied over the Wigner plane [13]

$$B_p(x, x') = C_1 \exp\left[i\pi\left(\frac{x^2 + x'^2}{\lambda f_1 \tan\phi}\right) - 2i\pi\left(\frac{x x'}{\lambda f_1 \sin\phi}\right)\right] \qquad (14)$$

Note that w is the coefficient that connects the two interpretations:

$$w = \sqrt{\frac{\lambda f_1}{\pi}} \qquad (15)$$

Careful examination of the expression of Eq. 14 shows that the FRT is a localized transformation. When one says localized, in this context, it means that the input function is actually multiplied by a space window as is done in the Gabor transform [14]. In the FRT case the space window is a phase window (the chirp phase function $\exp(i\pi\frac{x^2+x'^2}{\lambda f_1 \tan\phi})$) and not an amplitude window as in the original Gabor transform. In a chirp function one can notice that as the distance from the origin increases the spatial frequency increases as well and eventually the spatial frequency becomes so high that while calculating the integral of Eq. 12, we have under sampling case. As a result we lost the higher frequencies and the phase window is equivalent to an amplitude window.

2.3. Properties of the FRT

- Linearity:

  The FRT of a linear combination of two input functions u_1 and u_2 behave according to the definition of linear systems. c_1 and c_2 are constants.

$$\mathcal{F}^p[c_1 u_1(x) + c_2 u_2(x)] = c_1\{\mathcal{F}^p[u_1(x)]\} + c_2\{\mathcal{F}^p[u_2(x)]\} \quad (16)$$

- Continuity:

  Two FRTs with different orders p_1 and p_2 yield the following theorem:

$$\mathcal{F}^{c_1p_1+c_2p_2}[u(x)] = \mathcal{F}^{c_1p_1}[\mathcal{F}^{c_2p_2}(u(x))] = \mathcal{F}^{c_2p_2}[\mathcal{F}^{c_1p_1}(u(x))] \quad (17)$$

- Parseval's theorem:

$$\int_{-\infty}^{\infty} |u_0(x_0)|^2 dx_0 = \int_{-\infty}^{\infty} |u_p(x_p)|^2 dx_p \quad (18)$$

- Shift theorem:

  If the input object is shifted by the amount of a, then its FRT yields:

$$u_p(x;a) = \{\mathcal{F}^p[u_0(x_0+a)]\} = \exp[i\pi a \sin\phi(2x + a\cos\phi)] \cdot u_p(x + a\cos\phi) \quad (19)$$

- Scaling theorem.

  If the input object is scaled by the factor of a, then its FRT yields:

$$u_p(x;a) = \{\mathcal{F}^p[u_0(ax_0)]\} = \frac{\Psi}{a} u_{\hat{p}}\left(\frac{x}{\sqrt{\sin^2\phi(a^2 - \frac{1}{a^2}) + \frac{1}{a^2}}}\right) \quad (20)$$

where

$$\hat{p} = \frac{2}{\pi}\tan^{-1}\left[a^2\tan\left(\frac{\pi p}{2}\right)\right] \quad (21)$$

$$\Psi = \exp\left[\frac{i\pi x^2(a^4-1)}{a^4\tan\phi + \frac{1}{\tan\phi}}\right] \quad (22)$$

3. FRACTIONAL CORRELATION

In several pattern recognition applications the shift invariance property within all of the input plane is not necessary and even disturbs. An example is the case where the object is to be recognized only when its location is inside a certain area and rejected otherwise, e.g. a passport with a picture that should appear only at the upper right area. Several approaches for obtaining such space variance detection have been suggested [15]-[20]. A related solution is the tool coined fractional correlation (FC) which is based upon the fractional Fourier transform [21, 22]. The FC operation allows to control the amount of shift variant property of the correlation. This property is based on the shift variance of the FRT and it is more significant for the fractional orders of $p \approx 0 + 2N$ and less for $p \approx 1 + 2N$ (N is any integer).

In contrast to a solution of using an appropriate input pupil which is open in the desired location, the FC does not require any additional equipment for its optical implementation. An additional example for the necessity of the FC is the case where the recognition should mainly be based on the central pixels and less on the outer pixels (for instance in systems whose spatial resolution is improved in the central pixels, and thus the central region of pixels is more reliable for the recognition process). An important application for the FC might be the detection of localized objects using a single cell detector, eliminating the need for a CCD array detector.

The algorithm for performing a FC consists of obtaining the product of the fractional transforms of the distributions to be correlated, rendering a last FRT to obtain the final result. Analytically, the operation of FC of an input function, $f(x)$, with a reference pattern, $g(x)$, is defined as follows:

$$C_{p_1,p_2,p_3}(x') = \mathcal{F}^{p_3}\{\mathcal{F}^{p_1}\{f(x)\}\mathcal{F}^{p_2}\{g(x)\}\} \quad (23)$$

Where p_1, p_2, p_3 are the orders of the FRTs to perform, in principle arbitrary. Due to various reasons, detailed in Ref. [21], the most obvious choice is:

$$p_1 = p \qquad p_2 = -p \qquad p_3 = -1 \quad (24)$$

with p ranging from 0 to 1. In this case, if the input coincides with the reference object, a perfect phase matching between object and reference FRTs in the fractional domain is obtained. The inverse Fourier transform will just focus the resulting plane wave.

In order to build optically a FC, instead of preparing a full setup containing two lenses and free propagations, the object is illuminated with a converging beam (seen in Fig. 2) [23].

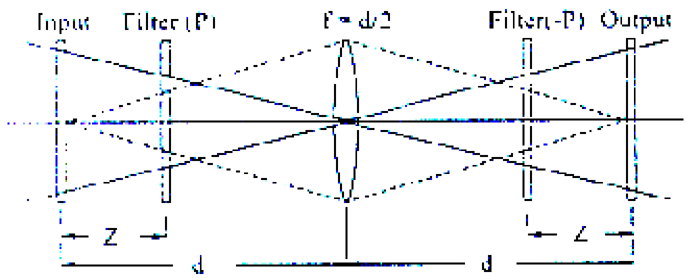

Figure 2: Experimental setup for obtaining the fractional correlation.

This permits the change of the convergence phase factor, multiplying the object, by displacing it along the optical axis. The matching between the distance object-filter and the convergence of the beam may produce any desired order and scaling factor. Hence, this approach is more convenient for the experimenter, as the exact sizes of the input and filter transparencies are often not precisely determined. This is especially important for the case of using SLMs for implementing the filter. As the FRT is not exact there will be a quadratic phase factor multiplying the output plane. It means that the correlation plane will be displaced along the optical axis.

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