APPLICATIONS OF THE FRACTIONAL FOURIER TRANSFORM TO FILTERING, ESTIMATION AND RESTORATION

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ABSTRACT

The fractional Fourier transform is more general and flexible than the ordinary Fourier transform, but its optical and digital implementation is just as efficient. This underlies its potential for generalizations and improvements in every area of digital and optical signal processing. Here we consider applications of the transform to filtering, estimation and restoration. We see that the use of fractional Fourier transform based filtering configurations allow one to flexibly trade off between cost and accuracy in these applications.

1. INTRODUCTION

In many applications of digital and optical signal processing, it is desired to implement general linear systems of the form $g(u) = \int H(u,u')f(u')\,du'$. Such systems take the form of a matrix-vector product when discretized: $g_k = \sum_{n=1}^{N} H_{kn} f_n$ or $\mathbf{g} = \mathbf{Hf}$. This may either represent a system which is inherently discrete or may constitute an approximation of a continuous system. Digital implementation of such general linear systems takes $O(N^2)$ time. Common single-stage optical implementations, such as optical matrix-vector multiplier architectures or multi-facet architectures [1] require an optical system whose space-bandwidth product is $O(N^2)$.

The output of a shift-invariant (convolution) system characterized by the impulse response $h(u)$ is related to the input by the relation $g(u) = \int h(u-u')f(u')\,du'$ whose discrete form is $g_k = \sum_{n=0}^{N-1} h_{k-n} f_n$, which is again a matrix-vector multiplication, but this time with the matrix being of a special form. Digital implementation of such shift-invariant systems takes $O(N \log N)$ time (by using the FFT) since they correspond to convolution in the time or space domain and multiplication with a filter function in the Fourier domain. Optical implementation requires an optical system whose space-bandwidth product is $O(N)$.

Due to the intrinsic nature of some problems , convolution-type systems are fully adequate. In fact, in some cases, they represent the optimal choice out of all linear systems. For example, the optimal linear estimation filter (Wiener filter) for time-invariant distortions and stationary signals turns out to be of the convolution form. However, in other cases, the use of shift-invariant systems is either totally inappropriate or at best a crude approximation which is employed only because of its significantly lower digital or optical implementation cost. This is not surprising given the fact that shift-invariant systems are a much more restrictive class than general linear systems, which is evident upon noting that general linear systems have N^2 degrees of freedom whereas shift-invariant systems have only N.

We may think of shift-invariant systems and general linear systems as representing two extremes in a cost-accuracy tradeoff. Sometimes use of shift-invariant systems may be inadequate, but at the same time use of general linear systems may be overkill and prohibitively costly. In such situations where both extremes are unacceptable, or simply when we desire greater flexibility in trading off between cost and accuracy, it would be desirable to be able to interpolate between these two extremes. There may be many ways of achieving this. In this paper we will consider multi-stage (repeated) filtering, multi-channel (parallel) filtering and filtering circuits in fractional Fourier domains. Common single-stage Fourier-domain filtering is shown in Fig.1a. The dual of this operation is single-stage time-domain filtering, and is shown in Fig.1b. Fig.1c depicts single-stage filtering in the ath order fractional Fourier domain. This filtering configuration generalizes the time and Fourier-domain filtering configurations and is discussed in [2, 3, 4] together with some applications.

The ath order fractional Fourier transform $\mathcal{F}^a$ is the generalization of the ordinary Fourier transform, such that $a = 1$ corresponds to the ordinary Fourier transform and $a = 0$ corresponds to the identity op-

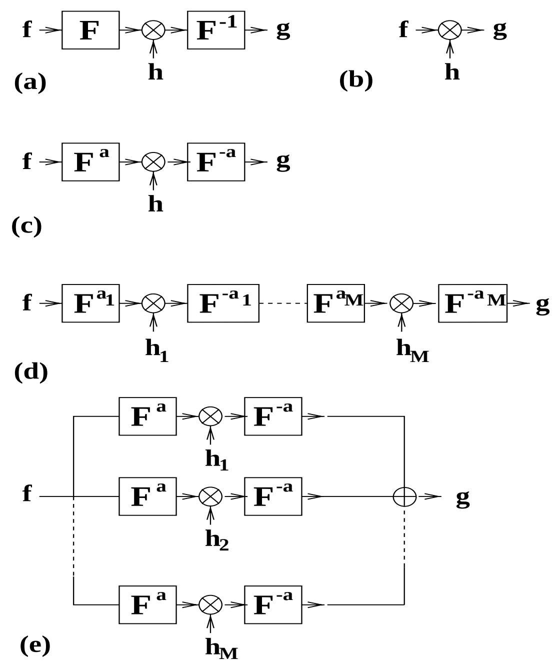

Figure 1: Single-stage filtering in the Fourier domain (a), the time domain (b), and the ath order fractional Fourier domain (c). Multi-stage (repeated) filtering (d). Multi-channel (parallel) filtering (e).

eration [5, 6, 8]. Thus, when $a = 1$, the filtering scheme in Fig. 1c corresponds to ordinary Fourier domain filtering (shift-invariant or convolution-type systems). When $a = 0$, it corresponds to direct multiplication by $h(u)$ in the time domain (Fig. 1b). The costs of both digital and optical implementations of fractional Fourier domain filtering are the same as that of ordinary Fourier domain filtering [2, 11]. We refer the reader to [5, 6, 7] for a more detailed introduction to the fractional Fourier transform together with some of its applications.

2. MULTI-STAGE AND MULTI-CHANNEL FILTERING

The multi-stage (repeated) filtering scheme (Fig. 1d) first suggested in [5, 9] consists of M single-stage blocks connected consecutively or in series. The input is first transformed into the a_1th domain where it is multiplied by a filter $h_1(u)$. The result is then transformed back into the original domain [13, 14]. This process is repeated M times. [1] (The back transform of stage k with order a_k, may be combined with the forward transform of stage $k+1$ with order a_{k+1}, resulting in a single transform of order $a_{k+1} - a_k$. Thus the system consists of multiplicative filters sandwiched between fractional transform stages of order $a'_k = a_{k+1} - a_k$.)

The multi-channel filter structure, on the other hand, consists of M single-stage blocks in parallel [16, 17]. For each channel k, the input is transformed to the a_kth domain, multiplied with a filter $h_k(u)$, and then transformed back (Fig.1e). (More generally, we may choose not to back transform, or to transform to some other domain.)

Let h_{j_n} denote the nth sample of the jth filter $h_j(u)$, and $\mathbf{\Lambda}_j$ denote the diagonal matrix whose elements are equal to those of the vector $\mathbf{h}_j = [h_{j_0}\ h_{j_1}\ \ldots\ h_{j_{N-1}}]^{\mathrm{T}}$. Then, the output vectors $\mathbf{g}$ are related to the input vectors $\mathbf{f}$ according to the relations[2]

$$\mathbf{g} = \left[\mathbf{F}^{-a_M}\mathbf{\Lambda}_M \ldots \mathbf{F}^{a_2-a_1}\mathbf{\Lambda}_1\mathbf{F}^{a_1}\right]\mathbf{f} = \mathbf{T}_{\mathrm{ms}}\mathbf{f}, \tag{1}$$

$$\mathbf{g} = \left[\sum_{k=1}^{M}\mathbf{F}^{-a_k}\mathbf{\Lambda}_k\mathbf{F}^{a_k}\right]\mathbf{f} = \mathbf{T}_{\mathrm{mc}}\mathbf{f}, \tag{2}$$

where $\mathbf{F}^{a_j}$ represents the discrete a_jth order fractional Fourier transform matrix [11, 18] and, $\mathbf{T}_{\mathrm{ms}}$ and $\mathbf{T}_{\mathrm{mc}}$ correspond to the overall kernel of the multi-stage and multi-channel configurations respectively. From the above equations we see that overall kernel $\mathbf{T}_{\mathrm{mc}}$ of the multi-channel filtering structure depends linearly on filter-coefficients h_{j_n} while $\mathbf{T}_{\mathrm{ms}}$ depends nonlinearly. Thus in a given application, the optimal filter coefficients may be found analytically in the multi-channel case [16], whereas they can be found by using iterative algorithms in the multi-stage case [13].

We used M single-stage fractional Fourier domain filters as building blocks to construct the multi-stage and multi-channel configurations in fractional domains. For the multi-stage case, they are combined in series. For the multi-channel case, they are combined in parallel. In analogy to circuit theory, we can further generalize these configurations to speak of filtering circuits in fractional Fourier domains [16, 13]. An example of such a filtering circuit is shown in Fig. 2.

The repeated and parallel configurations have at most $MN+M$ degrees of freedom. Their digital implementation will take $O(MN \log N)$ time since the fractional Fourier transform can be implemented in $O(N \log N)$ time [11]. Optical implementation will require an M-stage or M-channel optical system, each with space-bandwidth product N [12]. We see that these configurations lie (interpolate) between general linear

[1] It has been shown in [10] that, by modifying the filters $h_j(u)$ appropriately, the repeated configuration can be reduced to one involving only ordinary Fourier transforms. However, the modified filters often exhibit oscillatory behavior so that this reduction is not necessarily beneficial in practice.

[2] The extension to two-dimensions and/or rectangular matrices is straightforward.

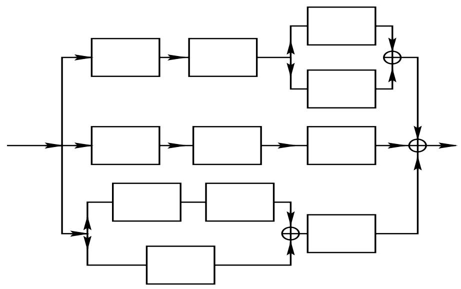

Figure 2: Each block corresponds to figure 1c.

systems and shift-invariant systems both in terms of cost and flexibility. If we choose M to be small, cost and flexibility are both low. If we choose M larger, cost and flexibility are both higher. In between, these systems give us considerable freedom in trading off efficiency and flexibility for each other, the latter which will translate into a better approximation and greater accuracy in most applications. $M = 1$ corresponds to single-stage filtering. As M approaches N, the number of degrees of freedom of the repeated filtering configuration approaches that of a general linear system.

The important point is that increasing M gives us greater flexibility and will allow us to realize a broader class of linear systems, or put in a different way, to better approximate a given linear system. In other words, the capabilities of an M-stage system can be characterized in two ways. First, for a given value of M, we can realize a certain subset of all linear systems exactly (or to some other specified degree of accuracy). As M increases, the subset in question becomes larger and larger. Second, and perhaps more useful, is to consider the problem of approximating a given linear system. For a given value of M, we can approximate this system with a certain degree of accuracy (or error). For instance, a shift-invariant system can be realized with perfect accuracy with $M = 1$. In general, there will be a finite accuracy for each value of M. As M is increased, the accuracy will usually increase (but never decrease). Thus, in the context of a particular application or problem, we can seek the minimum value of M which results in the desired accuracy, or the highest accuracy (or minimum error) that can be achieved for a given value of M. Of course, this amounts to seeking the best performance for given cost, or least cost for given performance. Such cost-performance points are referred to as Pareto optimal cost-performance combinations. The locus of such Pareto optimal points constitutes the cost-performance tradeoff curve.

The filtering configurations introduced above may find many applications in digital and optical signal processing, such as in signal recovery and estimation, system synthesis, and signal synthesis. In the recovery and estimation applications the aim is to recover the desired signal or system from the observed degraded data. This problem is well known to be an ill-posed problem. The general linear solution to this problem is known. However, the optical and digital implementation of this direct solution would require very high cost. For this reason, fractional Fourier domain filtering configurations which have efficient implementations may be used instead [13, 16, 14, 17].

In the synthesis applications our aim is to synthesize a desired input or output signal, or a desired system. The problem of approximately synthesizing a general linear system arises when we want to implement that system efficiently. If we can approximately synthesize a system in terms of a few number of other systems with efficient implementation algorithms, then we can reduce the cost considerably. The fractional Fourier domain filtering configurations suggest a flexible way of synthesizing general linear systems efficiently [13, 15, 16, 17].

An example of the signal synthesis problem arises in optics when we want to synthesize a desired optical wave from a given optical wave both of which are characterized by their statistical properties. These statistical properties are in general in the form of second-order statistics, and are given by mutual intensity functions in optics, and this leads to non-linear quadratic equations in most of the cases. By introducing a positive-square root representation we can reduce this nonlinear quadratic problem to a linear one, and then we can use the filtering configurations introduced above to efficiently synthesize the desired output optical field characterized by its mutual intensity function from a given optical source [3, 16, 13].

The proposed system can be used in a given application in one of two distinct ways, which we now distinguish. (i) Starting with a signal restoration, recovery, or reconstruction problem, we determine the optimal linear estimation or reconstruction matrix using any models and methods considered appropriate. Or, we may simply be given a matrix $\mathbf{H}$ to multiply input vectors $\mathbf{f}$ with. Then, we seek the transform orders a_j and filters $\mathbf{h}_j$ such that the resulting matrix $\mathbf{T}$ (as given by (1) or (2)) is as close as possible to $\mathbf{H}$ according to some specified criteria. (ii) We take (1) or (2) as a constraint on the form of the linear estimation or reconstruction matrix to be employed. Given a specific optimization criteria, such as minimum mean-square error, we find the optimal values of a_j and $\mathbf{h}_j$ such that the given criteria is optimized.

3. EXAMPLES

In this section we consider the application of the introduced filtering configurations to signal restoration. In the signal restoration problems, the aim is to recover or estimate the signals which are degraded by a known distortion and/or by noise. A commonly used observation model is

$$\mathbf{g} = \mathbf{G}\mathbf{f} + \mathbf{n}, \tag{3}$$

where $\mathbf{G}$ is the linear system that degrades the desired signal (may be an image) f, and n is an possibly additive noise term. We here propose the use of fractional Fourier domain filtering circuits to recover the desired signal so that our problem is to minimize the error

$$\sigma_e^2 \;=\; E\left[\|\mathbf{f} - \mathbf{T}\mathbf{g}\|^2\right]$$

where $\mathbf{T}$ is in the form of either $\mathbf{T}_{\mathrm{ms}}$, $\mathbf{T}_{\mathrm{mc}}$ or filtering circuits.

As an example, we consider the restoration of images blurred by a space-variant point spread function. The point spread function is local and Gaussian in shape and its width changes slightly with position. For the single stage case the normalized error between the restored image and original image turns out to be 22%. The multi-stage case results in error of 10%, and 3% for $M = 3$ and $M = 5$ filters respectively. The resulting error would be 9% and 5% for the multi-channel case with $M = 3$ and $M = 5$ filters. Figure 3 shows the original, degraded and restored images.

We also investigate the use of fractional Fourier domain filtering circuit concept in the above image restoration example. We consider the filtering circuit which consists of two branches each with two filters. The fractional domains are chosen as $a_1 = 0.25$, $a_2 = 0.5$ for the upper branch and $a_3 = 0.75$, $a_4 = 1$ for the lower branch. The overall operator representing this circuit is given by,

$$\mathbf{T}_{\mathrm{fc}} = \sum_{k=1}^{2} \mathbf{F}^{-a_{2k}} \mathbf{\Lambda}_{2k} \mathbf{F}^{a_{2k}-a_{2k-1}} \mathbf{\Lambda}_{2k-1} \mathbf{F}^{a_{2k-1}} \tag{4}$$

where $\mathbf{\Lambda}_l$ is a diagonal matrix whose diagonal consists of the filter vector $\mathbf{h}_l$ as before. In order to solve for the optimal filter functions we modify the iterative algorithm suggested in [13] so that we first initialize the filters $\mathbf{h}_1$ and $\mathbf{h}_3$ to vectors consisting of 1s and then solve for the optimal filter vector $\mathbf{h}_2$ and $\mathbf{h}_4$ [16]. We then take the solutions for $\mathbf{h}_2$ and $\mathbf{h}_4$ as constants and solve for the optimal filter vectors $\mathbf{h}_1$ and $\mathbf{h}_3$. We continue to apply this procedure iteratively until the error does not change significantly as compared to the previous step. The normalized error turns out to be 2%.

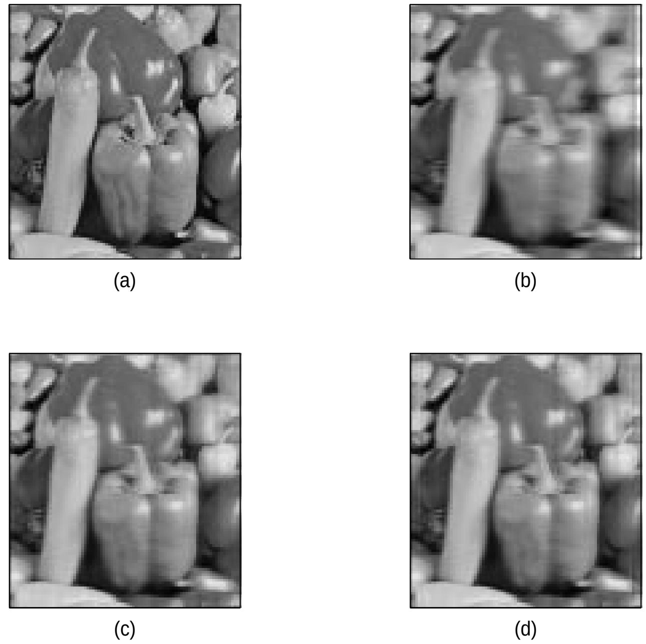

Figure 3: Original image (a). Blurred image (b). Restored by repeated filtering ($M = 5$) (c). Restored by multi-channel filtering ($M = 5$) (d).

This figure is slightly better than the figures obtained by the multi-stage and the multi-channel filtering configurations with 5 filters.

We next consider the problem of recovering a signal consisting of multiple chirp-like components buried in white Gaussian noise with a signal-to-noise ratio = 0.1. We assume that the signal consists of 6 chirps with uniformly distributed random amplitudes and time shifts, and that the chirp rates are known with a $\pm 5\%$ accuracy. With approach (i) described at the end of previous section, the multi-channel configuration results in a mean-square error of 5.2% with $M = 6$. With approach (ii), the same number of filters results in an estimation error of 2.6%.

4. DISCUSSION AND CONCLUSION

The repeated and multi-channel configurations may be based on other transforms with fast algorithms, instead of the fractional Fourier transform. For instance, the three-parameter family of linear canonical transforms may be used. Concentrating on (2), the essential idea is to approximate a general linear operator as a linear combination of operators with fast algorithms. If an acceptable approximation can be found with a value of M which is not too large, the computational burden can be significantly reduced.

Naturally, the number of stages and filters required to attain a given accuracy will be smaller for matrices exhibiting greater regularity or other more subtle forms of intrinsic structure. In such cases, direct implementation of the matrix-vector product is clearly inefficient. The regularity or structure inherent in a given matrix can be exploited on a case by case basis through ingenuity or invention; most sparse matrix algorithms and fast transform algorithms are obtained in this manner. In contrast, our method provides a systematic way of obtaining an efficient implementation which does not require ingenuity on a case by case basis. This approach would be especially useful when the regularity or structure of the matrix is not simple or is not expressed symbolically or when we are presented with a specific matrix in numerical form for which no easily discernible regularity or structure is apparent.

A distinct circumstance in which the method may be beneficial, even when a strong intrinsic structure does not exist, is when it is sufficient to compute the matrix-vector product with limited accuracy. This may be the case when some other component or stage of the overall system limits the accuracy to a lower value anyway, or simply when the application itself demands limited accuracy.

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