DESIGN OF DELTA-FILTERS USING LINEAR PROTOTYPES

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ABSTRACT

In this paper we introduce a class of nonlinear filters, whose impulse and frequency responses are expressed in terms of delta-functions. Since such filters named delta-filters are completely defined by univariate function, the design technique based on linear filter prototype can be used. We consider possible types of the delta-filters and their properties. A simple example demonstrating the proposed design technique is also included in the paper.

1. INTRODUCTION

As opposed to designing of linear filters, which is based on approximation of a given magnitude characteristic of an ideal filter (for example, low-pass or high-pass), a synthesis of nonlinear filters does not allow so simple and evident treatment. In the nonlinear signal filtering, spectrum transformation is a more sophisticated process resulting in intermodulations and other frequency interactions. At the same time using these more rich nonlinear phenomena just allows us to solve tasks linear filtering fails to do [1].

To describe nonlinear filtering effects in the frequency domain generalized frequency responses have been used [2]. As these functions are multidimensional, it is rather difficult to apply them for nonlinear filter design. Under some restriction on a class of input signals, for example, focusing solely on sine wave signals, efficient methods for quadratic filter design have been proposed [3, 4].

In a class of polynomial (Volterra) filters, it is possible to extract filters with kernels expressed in terms of delta-functions in the frequency or time domains. These filters named delta-filters are powerful enough to resolve a lot of tasks in practice. Definition of the delta-filters responses by one-dimensional slices enables to reduce a problem to design of a linear prototype filter and make use of well-known methods of linear filter design [5]. Besides, delta-filters are considerably easier to implement than common Volterra filters.

In this papers we shall present theoretical basis of the delta-filtering and demonstrate the design technique by using sine wave parameter estimation as an example.

2. FREQUENCY DOMAIN REPRESENTATION

A frequency response of an M-th order polynomial filter may be characterized with transfer functions (kernels in the frequency domain) $H_1(\omega), H_2(\omega_1,\omega_2), \ldots, H_M(\omega_1,\ldots,\omega_M)$ and defined as summation

$$Y(\omega) = \sum_{m=1}^{M} Y_m(\omega) \tag{1}$$

of a linear component $Y_1(\omega) = H_1(\omega)X(\omega)$ and nonlinear components $Y_m(\omega)$ ($m > 1$) which are given by

$$Y_m(\omega) = \frac{1}{(2\pi)^{m-1}} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} Y_m(\omega_1,\ldots,\omega_m)\delta\left(\omega - \sum_{i=1}^{m}\omega_i\right) d\omega_i \tag{2}$$

where $Y_m(\omega_1, \ldots, \omega_m) = H_m(\omega_1, \ldots, \omega_m)X(\omega_1)\cdot\ldots\cdot X(\omega_M)$.

For simplification of the frequency analysis of nonlinear systems, generalized describing functions have been introduced in [7]. These functions are defined only on the input frequency range and hence can not be used to describe of nonlinear effects connected with new frequencies generation.

In order to overcome this limitation, let's introduce some new concepts [8]. At first, we consider a spectrum of n-th power of an input signal defined by

$$X_m(\omega) = S_m\left[X(\omega)\right] = \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \delta\left(\omega - \sum_{i=1}^{m}\omega_i\right) \prod_{i=1}^{m} X_i(\omega_i) d\omega_i \tag{3}$$

This function contains all frequency components of m-th order nonlinearity. We shall call $X_m(\omega)$ *a nonlinear input spectrum*, and $Y_m(\omega)$ given by (2) – *a nonlinear output spectrum* of the order m.

To characterize the relation between these spectra at a frequency ω, let's also define a function

$$G_m(\omega, X(\omega)) = \frac{Y_m(\omega)}{X_m(\omega)} \tag{4}$$

for $X_m(\omega) \neq 0$. If $m = 1$, this function is the frequency response of a linear filter. For $m \geq 2$, this one depends on $X(\omega)$ and may be considered as a quasi-linear description of m-th order nonlinearity. As follows from (2) – (4) the function $G_m(\omega, X(\omega))$ describes a relationship between frequency components of the spectra $X_m(\omega)$ and $Y_m(\omega)$ derived by integration on hyperplanes $\omega_1 + \ldots + \omega_m = \omega$. Therefore, function $G_m(\omega, X(\omega))$ may be regarded as *integral frequency response* of the m-th order.

Now using (4) we can rewrite the frequency response $Y(\omega)$ of M-th order polynomial filter as follows

$$Y(\omega) = G_1(\omega)X(\omega) + \sum_{m=2}^{M} G_m(\omega, X(\omega))X_m(\omega) .$$

This series is an equivalent representation of the polynomial filter in the form similar to Hammerstein's model with input dependent frequency responses.

3. FILTER DESCRIPTION VIA KERNEL SLICES

In contrast to linear case, for synthesis of nonlinear filters with given frequency properties, it is necessary beforehand to define a class of input signals. In order to analyze possible output frequency component of the filter, at first the input nonlinear spectrum can be derived. It allows one to choose the order and structure of the filter suitable for the given task. Then, requirements to the integral frequency response are formed, and filter coefficients are calculated.

In order to simplify the analysis and synthesis of filters in the frequency domain, a sine wave input may be selected. In practice, by using even a single sinusoid it is possible to get qualitative picture of filtering process, but also to gain insight into choice of a structure and parameters of a nonlinear filter being designed.

For an important subclass of quadratic filters with input signal $x(n) = A_0 + A_1\cos\lambda n$, Eqs. (3) and (4) can be rewritten as

$$X_2(\omega) = 2\pi\left[(A_0^2 + 2A_1^2)\delta(\omega) + 2A_0A_1\delta(\omega-\lambda) + A_1^2\delta(\omega-2\lambda)\right],$$

$$G_2(\omega, A_0, A_1) = \begin{cases} \dfrac{A_0^2 H_2(0,0) + 2A_1^2 H_2(-\lambda,\lambda)}{A_0^2 + 2A_1^2}, & \omega = 0; \\ H_2(0,\lambda), & \omega = \lambda; \\ H_2(\lambda,\lambda), & \omega = 2\lambda. \end{cases} \quad (5)$$

It should be noted that the integral frequency response is defined at the frequencies $\omega = 0$, $\omega = \lambda$, $\omega = 2\lambda$ such that $X_2(\omega) \neq 0$ and can be represented by the three components $\{G_{2,0}(\lambda)\delta(\omega), G_{2,1}(\lambda)\delta(\omega-\lambda), G_{2,2}(\lambda)\delta(\omega-2\lambda)\}$ expressed in terms of one-dimensional slices of the two-dimensional transfer function $H_2(\omega_1, \omega_2)$.

For m-th order, the integral frequency response $G_m(\omega)$ of the nonlinear filter for the sinusoidal input is the following set $\{G_{m,0}(\lambda)\delta(\omega), G_{m,1}(\lambda)\delta(\omega-\lambda), ..., G_{m,m}(\lambda)\delta(\omega-m\lambda)\}$ defined by the one-dimensional slices of the m-dimensional transfer function $H_m(\omega_1, ..., \omega_m)$.

The slices specify a contribution of different harmonics to the output signal and, therefore, may be used to provide the given filter performance. For example, to suppress DC and a frequency component at ω=2λ in the output signal of the quadratic filter the following condition has to be satisfied:

$$H_2(0,0) = H_2(-\lambda,\lambda) = H_2(\lambda,\lambda) = 0.$$

The requirements produced to the kernel slice in the frequency domain can be transformed into equivalent ones to the nonlinear impulse response which are required for implementation of the filter to be designed.

Let the slice of the of the m-th order frequency kernel be

$$H_m(\underbrace{\lambda,\ldots,\lambda}_{a},\underbrace{0,\ldots,0}_{b},\underbrace{-\lambda,\ldots,-\lambda}_{c}) \quad (6)$$

where $a+b+c=m$. For brevity we shall designate (6) by $H_m(\lambda_{(a)}, 0_{(b)}, -\lambda_{(c)})$ and call it (a, b, c)-slice.

Proposition 1. *If a nonlinear impulse response $h_m(n_1, ..., n_m)$ satisfies the following condition*

$$\sum_{n_1,\ldots,n_m}\cdots\sum h_m(n_1,\ldots,n_m)\delta(n-n_1-\ldots-n_a+n_{a+1}+\ldots+n_{a+c}) = 0$$

for $\forall n$, then (a, b, c)-slice $H_m(\lambda_{(a)}, 0_{(b)}, -\lambda_{(c)}) \equiv 0$.

Similarly to the slice $H_m(\lambda_{(a)}, 0_{(b)}, -\lambda_{(c)})$ in the frequency domain we shall define (a, b, c)-slice in the time domain as

$$h_m(n_{(a)}, 0_{(b)}, -n_{(c)}) = h_m(\underbrace{n,\ldots,n}_{a},\underbrace{0,\ldots,0}_{b},\underbrace{-n,\ldots,-n}_{c}) .$$

As the slices depend on one argument, they can be characterized by the following univariate functions:

$$H_m(\lambda_{(a)}, 0_{(b)}, -\lambda_{(c)}) = \tilde{H}_m(\lambda) ,$$

$$h_m(n_{(a)}, 0_{(b)}, -n_{(c)}) = \tilde{h}_m(n) .$$

Proposition 2. *Let m-th order kernels be defined by their slices as*

$$H_m(\lambda_1,\ldots,\lambda_m) = (2\pi)^{m-1}\tilde{H}_m(\lambda_1)\prod_{i=2}^{a}\delta(\lambda_1-\lambda_i)\times$$

$$\times\prod_{k=1}^{b}\delta(\lambda_{a+k})\prod_{j=1}^{c}\delta(\lambda_1+\lambda_{a+b+j}),$$

$$h_m(n_1,\ldots,n_m) = \tilde{h}_m(n_1)\prod_{i=2}^{a}\delta(n_1-n_i)\prod_{k=1}^{b}\delta(n_{a+k})\times$$

$$\times\prod_{j=1}^{c}\delta(n_1+n_{a+b+j}) .$$

Then there exist the following relations in the terms of Fourier transform

$$H_m(\lambda_1,\ldots,\lambda_m) \Leftrightarrow h_m(n_1+\ldots+n_a-n_{a+1}-\ldots-n_{a+c}) ,$$

$$h_m(n_1,\ldots,n_m) \Leftrightarrow H_m(\lambda_1+\ldots+\lambda_a-\lambda_{a+1}-\ldots-\lambda_{a+c}) ,$$

$$\tilde{H}_m(\lambda) \Leftrightarrow h_m(n) , \qquad H_m(\lambda) \Leftrightarrow \tilde{h}_m(n) .$$

4. NONLINEAR DELTA–FILTERS

Let's consider a class of nonlinear filters with the impulse response determined on (a, b, c)-slice. Taking into account the symmetry of the impulse response $h_m(n_1, ..., n_m)$, this one can be expressed through discrete delta-function as

$$h_m(n_1,\ldots,n_m) = \left\{ h(n_1)\prod_{i=2}^{a}\delta(n_1 - n_i)\prod_{k=1}^{b}\delta(n_{a+k}) \times \right. \\ \left. \times \prod_{j=1}^{c}\delta(n_1 + n_{a+b+j}) \right\}_{sym} \qquad (7)$$

where $\{\bullet\}_{sym}$ stands for symmetrization by permutation of indices, involving $a!b!c!/m!$ terms.

We shall denote the nonlinear filter with impulse response (7) by $F_m^{(a,b,c)}[x(n)]$ and name it (a, b, c) delta-filter in the time domain.

According to proposition 2 the transfer function of the filter $F_m^{(a,b,c)}[x(n)]$ can be written as

$$H_m(\omega_1,\ldots,\omega_m) = \left\{ H(\omega_{i_1} + \ldots + \omega_{i_a} - \omega_{i_{a+1}} - \ldots - \omega_{i_{a+c}}) \right\}_{sym} .$$

Thus, the delta filters, being nonlinear, are completely characterized by one-dimensional functions $h(n)$ and $H(\omega)$, that essentially simplifies their design.

Example. Quadratic (1,0,1) delta-filter

$$y(n) = F_2^{(1,0,1)}[x(n)] = \sum_{i=-\infty}^{\infty} h(i)x(n+i)x(n-i)$$

with impulse response

$$h_2(n_1,n_2) = \frac{1}{2}\left[h(n_1)\delta(n_1+n_2) + h(n_2)\delta(n_1+n_2)\right]$$

defined on a diagonal $n_1 = -n_2$. Frequency properties of this filter are described by its transfer function

$$H_2(\omega_1,\omega_2) = \frac{1}{2}\left[H(\omega_1-\omega_2) + H(\omega_2-\omega_1)\right]. \qquad (8)$$

If the following condition holds

$$\sum_{i=-\infty}^{\infty} h(i) = 0\,, \qquad (9)$$

then $H_2(\omega, \omega) = H(0) = 0$, and the filter will suppress the second harmonic, pass DC and the first harmonic with transfer functions $H(2\omega)$, $H(\omega)$, respectively.

There are 3 types of the delta-filters of the second order, 5 types of the third order. Generally, for the m-th order nonlinearity, the amount of the possible types is given by $m + \lfloor m/2 \rfloor \lfloor (m+1)/2 \rfloor$ where $\lfloor m \rfloor$ denotes the integer part of the number m.

The (a, b, c) delta-filter of the order m with $b \neq 0$ can be factorized as

$$F_m^{(a,b,c)}[x(n)] = x^b(n) F_{m-1}^{(a,0,c)}[x(n)]. \qquad (10)$$

The impulse response of the $(a, 0, c)$ delta-filter in the right part of (10) is defined on m-dimensional diagonals crossing the origin. Therefore, we shall name such filter a *diagonal filter*. The diagonal filters are nonfactorable and represent some kind of base elements for construction of the complete set of the delta-filters. There are $\lfloor (m+2)/2 \rfloor$ diagonal filters of the m-th order.

In view of a time-frequency duality we can also introduce a class of *delta-filters in the frequency domain*. The frequency response $H_m(\omega_1, ..., \omega_m)$ of such filters differs from zero only on (a, b, c)-slice in the frequency domain and is given by

$$H_m(\omega_1,\ldots,\omega_m) = (2\pi)^{m-1}\left\{ H(\omega_1)\prod_{i=2}^{a}\delta(\omega_1 - \omega_i) \times \right. \\ \left. \times \prod_{k=1}^{b}\delta(\omega_{a+k})\prod_{j=1}^{c}\delta(\omega_1 + \omega_{a+b+j}) \right\}_{sym} .$$

The corresponding impulse response depends on sum/difference of arguments

$$h_m(n_1,\ldots,n_m) = \left\{ h(n_{i_1} + \ldots + n_{i_a} - n_{i_{a+1}} - \ldots - n_{i_{a+c}}) \right\}_{sym} .$$

Thus, each delta-filter $F_m^{(a,b,c)}[x(n)]$ in the time domain has its counterpart $\tilde{F}_m^{(a,b,c)}[x(n)]$ in the frequency domain.

In the frequency domain, it is also possible to extract a set of the factorable delta-filters, which can be expressed through the filters of the lower order as follows

$$\tilde{F}_m^{(a,b,c)}[x(n)] = S^b[x(n)]\tilde{F}_{m-1}^{(a,0,c)}[x(n)]$$

where $S[x(n)] = \sum_i x(n-i)$.

Here, the class of the unfactorable filters consists of $(a, 0, c)$ diagonal filters with the frequency response determined on m-dimensional diagonals in the frequency domain.

The implementation of the delta-filter is much simpler than that of the general polynomial filter. In the time domain, it is reduced to weighing by $h(i)$ of the input signal samples along (a, b, c)-slice according to the formula

$$y(n) = x^b(n)\sum_i h(i)x^a(n-i)x^c(n+i)\,.$$

In the frequency domain, weighing is carried out over the sums of sample products located on hiperplanes perpendicular to (a, b, c)-slice as follows

$$y(n) = \left(\sum_j x(n-j)\right)^b \sum_i h(i) \times$$

$$\times \sum_{\substack{i_1 \\ i_1+\ldots+i_{a+c}=i}} \ldots \sum_{i_{a+c}} x(n-i_1)\cdot\ldots\cdot x(n-i_a)x(n+i_{a+1})\cdot\ldots\cdot x(n+i_{a+c}) .$$

The variety of the delta-filters allows us to choose the filter most suitable for decision of the given task. Since any delta-filter is completely defined by one-dimensional impulse response $h(i)$, it is quite possible to reduce the delta-filter design to construction of a proper linear prototype filter.

5. DESIGN EXAMPLE

In order to demonstrate the usefulness and simplicity of delta-filter design using linear prototype filter, as an example, we now consider a problem of estimation of sine wave signal parameters.

Let's take the class of quadratic filters with a sinusoidal input signal $x(n) = A\sin\lambda n$. According to (5) the integral frequency response is given by

$$G_2(\omega) = \begin{cases} H_2(-\lambda,\lambda), & \omega = 0; \\ H_2(\lambda,\lambda), & \omega = 2\lambda. \end{cases}$$

The output spectrum involves DC and the second harmonic. It can be written as

$$Y(\omega) = G_2(\omega)X_2(\omega) = A^2\left[H_2(-\lambda,\lambda)\delta(\omega) + H_2(\lambda,\lambda)\delta(\omega-2\lambda)\right].$$

The desired output signal is DC which would be proportional to the parameter being estimated. To suppress an undesirable second harmonic, it is necessary that the slice $H_2(\omega,\omega)=0$. Varying the slice $H_2(-\omega,\omega)$, we can construct the quadratic filters with various properties. In particular, the following cases are possible:

(a) if $H_2(-\omega, \omega)=\alpha$ for $\omega \in [0, \omega_c]$, that corresponds to low-pass filtering with cutoff frequency ω_c, then the output signal is given by $y_1(n) = \alpha A^2$ and proportional to the input power;

(b) if $H_2(-\omega, \omega)=\beta\omega$ for $\omega \in [0, \omega_c]$, then the output signal is given by $y_2(n)=\beta A^2\omega$ and also proportional to the input frequency.

As an example we shall investigate the behavior of the FIR diagonal filter of the type (1,0,1) which is defined by nonlinear convolution

$$y(n) = \sum_{i=-N/2}^{N/2} h(i)x(n-i)x(n+i) . \qquad (11)$$

The frequency slices can be obtained from (8) as follows

$$H_2(-\omega,\omega) = H(2\omega) , \qquad H_2(\omega,\omega) = H(0) .$$

As mentioned before, $H_2(\omega, \omega)=0$ if (9) holds.

In order to obtain the output signal $y(n) = \beta A^2\omega$ in the frequency range $[0, \omega_c]$, as the prototype we have to use a filter with linear increasing frequency response $H(\varpi)$ in the range $[0, \varpi_c]$. It can be seen that the relationship between the cutoff frequencies ω_c and ϖ_c of the quadratic filter and the linear prototype filter is defined by

$$\varpi_c = 2\sqrt{2}\omega_c . \qquad (12)$$

At the same time in this class of the delta-filters, it is impossible to construct the filter whose output signal would be $y(n)=\alpha A^2$. In fact, the requirements $H_2(-\omega, \omega) = H(2\omega) = \alpha$ and $H_2(\omega, \omega) = H(0) = 0$ are incompatible.

At first sight, as the prototype filter a differentiator with the frequency response

$$H(\varpi) = \begin{cases} j\varpi, & |\varpi| \le \varpi_c; \\ 0, & |\varpi| > \varpi_c. \end{cases}$$

could be used. The relevant impulse response is given by

$$h(n) = \frac{1}{\pi}\left[\frac{\sin n\varpi_c}{n^2} - \frac{\omega_c \cos n\varpi_c}{n}\right].$$

As this function is antisymmetric, i. e. $h(-n) = -h(n)$, the output signal $y(n)$ of the (1,0,1) filter vanishes for any input signal $x(n)$. Therefore, it is impossible to use the differentiator as the prototype filter.

The decision can be found in a class of linear filters with frequency response

$$H(\varpi) = \begin{cases} |\varpi|, & |\varpi| \le \varpi_c; \\ 0, & |\varpi| > \varpi_c. \end{cases}$$

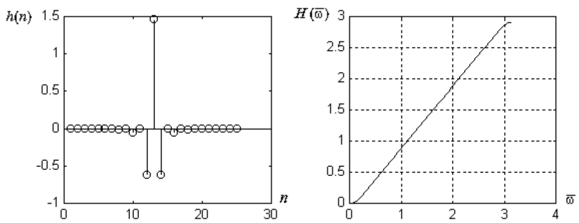
In this case, the impulse response is a symmetric function

$$h(n) = \frac{1}{\pi}\left[\frac{\varpi_c \sin n\varpi_c}{n} - \frac{1-\cos n\varpi_c}{n^2}\right].$$

For design of the linear prototype filter, a windowing technique with optimal Kaiser's window of length $N = 25$ has been used under the constraint (9). The impulse $h(n)$ and frequency $H(\varpi)$ responses of the wide-band prototype filter with maximum cutoff frequency $\varpi_c = \pi$ are shown in Fig. 1. The corresponding two-dimensional frequency response $H_2(\omega_1, \omega_2)$ of the quadratic diagonal filter is shown in Fig. 2. Contours of the response $H_2(\omega_1, \omega_2)$ in the frequency domain are lines perpendicular to a diagonal $n_1 = -n_2$ in the time domain along which the impulse response $h(n)$ of the prototype filter is arranged. According to (12) the cutoff frequency of the designed quadratic filter is equal to $\sqrt{2}\pi/4$.

The results of the sinusoidal signal filtering for step and linear frequency variation are depicted in Fig. 3. The experiment has shown that the output level depends quite linear on the input frequency up to the cuttoff frequency $\omega_c = \sqrt{2}\pi/4$. It is especially important that the estimation time is considerably less than a period of the sinusoid. That allows one to use this approach for a fast parameter estimation of low-frequency signals.

As it can be seen from Fig. 1(a) the impulse response $h(n)$ is localized in a small neighborhood of zero, and the main contribution is made by three samples: central and two lateral, which are approximately twice as small in magnitude. Leaving in convolution (12) only these terms, as a special case, we



(a) (b)

Figure 1. (a) impulse $h(n)$ and (b) frequency $H(\varpi)$ responses of the wide-band prototype filter.

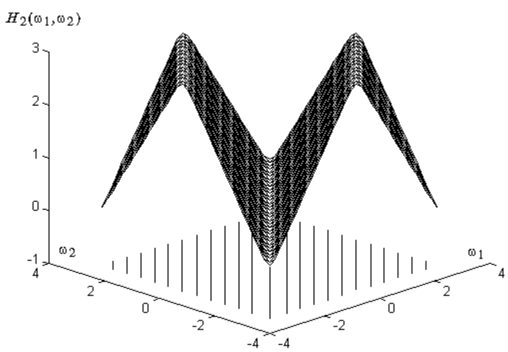


Figure 2. Frequency response of the quadratic (1,0,1) delta-filter.

obtain a well-known Kaiser's nonlinear operator [9]

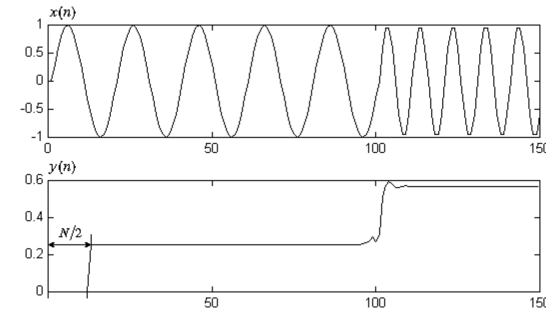
$$y(n) = x^2(n) - x(n-1)x(n+1)$$

with a kernel $H_2(\omega_1, \omega_2) = 1 - \cos(\omega_1 - \omega_2)$. For a sinusoidal input signal $x(n) = A\sin(\lambda n + \varphi)$, the output signal is DC $y(n) = A^2\sin^2\lambda$ ($\approx A^2\lambda^2$ for small values of λ). Thus, this operator can be used as a simple power estimator.

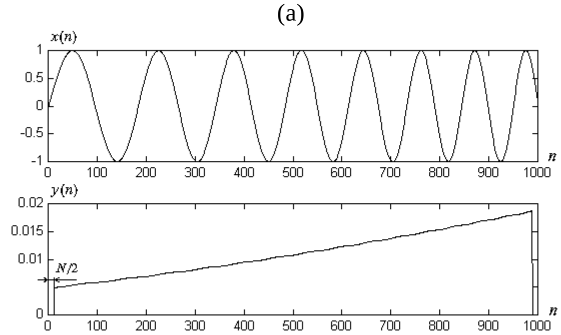
5. CONCLUSION

In this paper a subclass of polynomial filters with impulse or frequency responses expressed in terms of delta-functions is presented. The designing and implementation of these delta-filters is much easier than in a common case. At the same time they allow us to find a simple decisions in practice. For the delta-filter design, methods of linear filter design can be used. In this case, one-dimensional response of linear prototype filter is transformed into a multi-dimensional delta-filter response.

It is possible to enumerate all structures of the delta-filters in the frequency and time domains. Each delta-filter in the time domain corresponds to dual one in the frequency domain. Depending on a goal of design we can select the most appropriate type of delta-filter. As a rule, a obvious interpretation can be made to the delta-filter being designed. A simple example has been presented to show the usefulness of the proposed design technique.



(a)



(b)

Figure 3. Filtering results: (a) step frequency variation, (b) linear frequency variation.

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