CORRELATION PROPERTIES OF CASCADED RECURSIVE MEDIAN FILTERS

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ABSTRACT

In this paper we will present a series of experiments comparing the correlation characteristics of two nonlinear data smoothing methods. Firstly the Recursive Median sieve, a multiscale data-analysis system, implemented as a cascade of Recursive Median (RM) filters of increasing window lengths. Secondly a plain RM filter, applied directly to the original input signal. The point that we want to make, backed by a number of experiments, is that the RM filter is not in itself a reliable estimator of location, and it should not be used in data smoothing. As the cascading element in the structure of the sieve, however, the RM filter is very useful. The secondary aim of the paper is to discuss different methods of describing cross correlation characteristics.

1. INTRODUCTION

One of the problems arising from the use of median type filtering techniques is the phenomenon of streaking. This has been analysed by Bovik [3]. In image processing streaking can be identified as an effect that produces runs of equal values in the output. These runs have no visual correlate with the input. In 1D streaking produces effects like breakdown of the cross correlation function. These effects are examined here experimentally. Median type filters, at least the ones that we are considering, always produce one of the input samples at the output, a property not shared by linear systems. This can sometimes be considered useful, e.g. rounding errors are avoided, and non-pre-existing sample values are not introduced, however, problems like streaking can occur.

While Bovik in his paper concentrates on nonrecursive median filters, both 1D and 2D, we focus on Recursive Median (RM) filters, which include positive feedback. Although RM filters have been extensively used in applications, due to some of their agreeable properties, like robustness, effective noise suppression and idempotency, it is our conclusion that extreme care should be taken when RM filters are applied. In fact the 1D RM filter as a location estimate should always be applied in a cascade construction, the sieve, where the filters are strictly applied in the order of increasing window lengths. Recently new analytical results supporting this observation have been published, see Alliney [1] for a treatment of the Recursive Median sieve in the framework of regularisation theory, and Yli-Harja *et al.* [8] for an analysis of the sieve structure. For an illustration of the problem of streaking and the solution provided by the RM sieve, see Figure 1 in the appendix. It must be emphasised that actually the Datasieve is a much more general concept [5], and here we only consider the one dimensional self dual version of it.

2. EXPERIMENTS

We will compare two multiscale signal decomposition structures with the following experiments. Firstly the Recursive Median sieve, a multiscale data-analysis system, using a cascade of Recursive Median (RM) filters of increasing window lengths. Secondly a plain RM filter, with a filter window length equalling that of the last phase of the sieve, applied directly to the input signal.

We will investigate the properties of the cross correlation functions, and discuss the alternative ways of defining it. One such alternative is morphological correlation described by Maragos [7]. We will also propose a way of defining a useful measure of correlation, especially suited to signal processing problems involving median type filters, namely sample selection probability function. This correlation measure will estimate the amount of dependence caused by median type filters, independent of the shape of input distribution.

Thus in our experiments we will consider three correlation measures, namely linear cross correlation (L2-correlation), morphological correlation (L1-correlation) [7], and a measure based on exact equality of samples (EQ-correlation or the sample selection probability function). In the following we will introduce them briefly.

Let $f(n)$ be an arbitrary signal and $g(n)$ a pattern to searched from f. To find the best match, an error criterion such as *mean squared error* (MSE) can be minimised

$$E_2(k) = \sum_{n \in W} (f(n+k) - g(n))^2 . \tag{1}$$

Since $(a-b)^2 = a^2 + b^2 - 2ab$, minimising (1) equals maximising

$$\gamma_{fg}(k) = \sum_{n \in W} f(n+k) g(n), \tag{2}$$

yielding the classical (sum of products) linear correlation. Using the mean absolute error (MAE) criterion

$$E_1(k) = \sum_{n \in W} \left| f(n+k) - g(n) \right|, \tag{3}$$

and noticing that $|a-b| = a+b-2\min(a,b)$, we can define morphological correlation [7]

$$\mu_{fg}(k) = \sum_{n \in W} \min(f(n+k), g(n)). \tag{4}$$

Normalisation of this measure poses a problem, because the expected level, when *f* an *g* are totally independent, depends on the shape of the input distribution. In order to limit the scope of the paper, however, we show L1-correlation in unnormalised form.

Correlation based on measuring the number of exactly equal samples, the so called EQ-correlation can be defined by

$$\sigma_{fg}(k) = \sum_{n \in W} (f(n+k) = g(n)), \qquad (5)$$

where the result of sample-wise comparison is taken to be real 0 or 1.

EQ-correlation is severely handicapped in many situations by the requirement of exact matching. Still, in the case of Recursive Median filtering EQ-correlation proves to be useful because the output will always be one of the input samples, and there is no possibility of numerical inaccuracy. Also, while our experiments are implemented with Matlab using double variables, the probability of two independent and uniformly distributed random variables having the same value by accident, is very small. For example in our experiments, the machine epsilon, (minimum ε such that $(1+\varepsilon) > 1$), being $2.2204 \cdot 10^{-16}$, the probability for this to happen is negligible. This is, of course, assuming a good pseudo random number generator. Furthermore EQ-correlation is totally independent of the shape of the input distribution, as long as it has no point concentrations of probability. We can also interpret EQ cross correlation as a sequence of sample selection probabilities. Finally, EQ cross correlation manages to produce meaningful results in cases where L1 and L2 norm correlation measures fail, (observations 3 and 4).

The test signal employed was a sequence of 2000 independent and identically distributed samples. The distribution function was taken to be uniform, Gaussian or Laplacian with zero mean and unit variance. This signal was then fed to the sieve, implemented with cascaded RM-filters of increasing window length, see figure 2(a). The same input signal was also filtered with RM-filters of increasing window lengths, arranged into a parallel structure so that the input for each filter was the original signal, see figure 2(b). The three methods of measuring cross correlation were used to investigate the dependencies between the output signal at each scale, and the original signal and the signal at the previous scale. The results are illustrated in the appendix.

Finally are the experimental results statistically significant? This is demonstrated by repeating selected tests 50 times, and computing the standard deviation. These results are illustrated in figure 3.

3. CONCLUSIONS

We will next summarise the relevant conclusions in the form of observations.

Observation 1. L2- and L1-norm cross correlation produce effectively similar results. See figure 4.

Observation 2. For the cascaded RM-filter sieve all correlation measures produce easily detectable localised peaks, see figure 5.

Observation 3. For the parallel RM-filter structure L1- and L2-correlation vanishes for longer filter window lengths, figures 6a and 6b. This is due to massive streaking.

Observation 4. EQ-norm correlation measure, that is, the sample selection probability function, describes the behaviour of the parallel structure meaningfully, figure 6c. Qualitative interpretation is that the probability of a sample finding its way to the output from far behind is large.

Observation 5. Sample selection probability function, as well as other measures, for the sieve converges to a nearly symmetric function, when scale increases. This is illustrated by figure 5.

Observation 6. Correlation measured against the signal at the previous scale of the Recursive Median sieve, is seriously asymmetric only in the first stage, (3 point RM filter). This applies to all correlation measures, see figure 7. Recursive Median filters of longer window lengths, surprisingly, have nearly symmetric responses.

REFERENCES

[1] S. Alliney, "Recursive Median Filters of Increasing Order: A Variational Approach", *IEEE Transactions on Signal Processing*, vol. SP-44, no. 6, pp. 1346-54, Jun. 1996.

[2] O. Yli-Harja, J. Astola and Y. Neuvo, "Analysis of the properties of median and weighted median filters using threshold logic and stack filter representation", *IEEE Transactions on Signal Processing*, vol. SP-39, no. 2, pp. 395-410, Feb. 1991.

[3] A. C. Bovik, " Streaking in Median Filtered Images", *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. ASSP-35, no. 4, pp. 493-503, Apr. 1987.

[4] O. Yli-Harja, J. A. Bangham, R. Harvey, R. Aldridge, "Recursive Median Filtering in Constant Time", IEEE Transactions on Signal Processing, submitted.

[5] J. A. Bangham, P. Ling, R. Young, "Multiscale Recursive Medians, Scale-Space and Transforms with Applications Image Processing", *IEEE Transactions on Image Processing*, **5**(6), 1043-48 (1996).

[6] O. Yli-Harja, "Statistical Characterisation of Stack Filters", IEEE Workshop on Nonlinear Signal and Image Processing, Mackinaw Island, Michigan, USA, Sept. 1997.

[7] P. Maragos, "Morphological Correlation and Mean Absolute Error Criteria", *Proc. IEEE ICASSP-89*, Glasgow, Scotland, pp. 1568-71, 1989.

[8] O. Yli-Harja, J. A. Bangham, G. Cawley, R. Harvey, P.Koivisto and I. Shmulevich, “Implementing the Recursive Median Sieve with the 3-point Median Operation”, Signal Processing, submitted.

[9] O. Yli-Harja, "Formula for the Joint Distribution of Stack Filters", IEEE Signal Processing Letters, vol. 1, no. 9, pp. 129-130, Sept. 1994.

APPENDIX

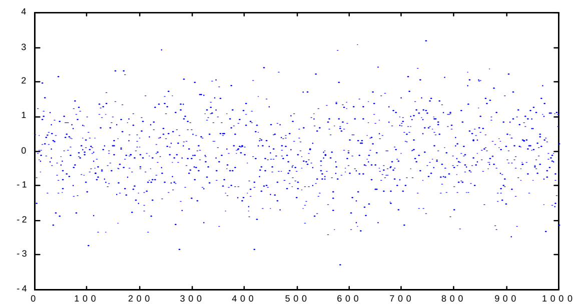

Figure 1. Illustration of the problem of streaking and the solution provided by the Recursive Median sieve. a) Original signal, normally distributed noise, 1000 samples.

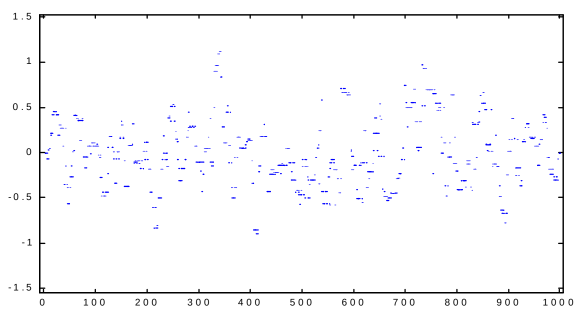

Figure 1b. Original signal filtered with the 13 point Standard Median filter. Minor streaking.

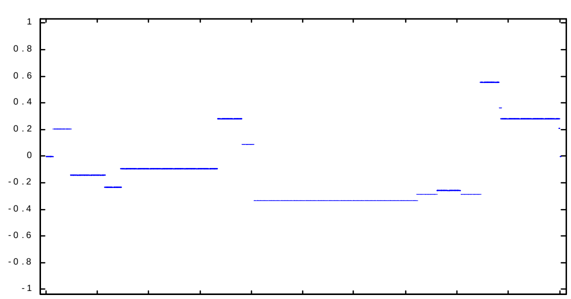

Figure 1c. Original signal filtered with the 13 point RM filter. Massive streaking.

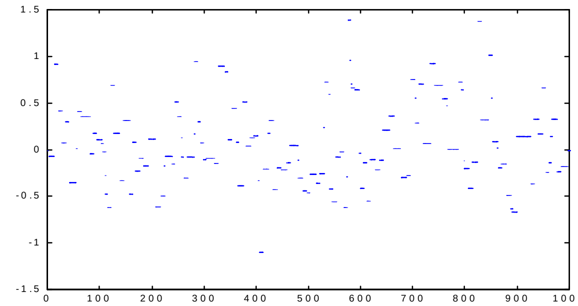

Figure 1d. RM sieve output at scale 6, (last filter applied is 13 point RM filter). Minor streaking.

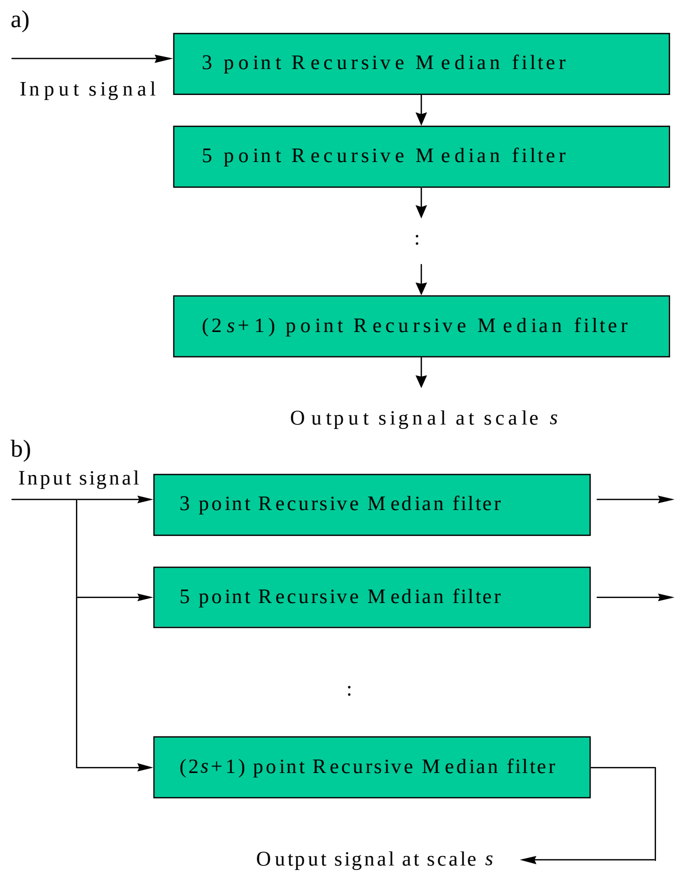

Figure 2. Illustration of the tested filtering structures. a) Cascade of Recursive Median filters with increasing window lengths, i.e. the 1D Recursive Median sieve. Parallel structure where the input for each RM filter is the original signal.

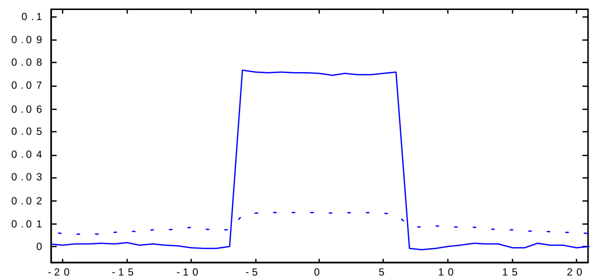

Figure 3. Selected tests repeated 50 times. All tests employ a 2000 point i.i.d. normally distributed test signal. This signal is filtered with the specified filter, correlation with the input signal is computed (with the specified method), and mean (solid line), and standard deviation (dotted line), are computed. a) 13 point Standard Median filter, L2-norm cross correlation. Compact support and symmetric dependency.

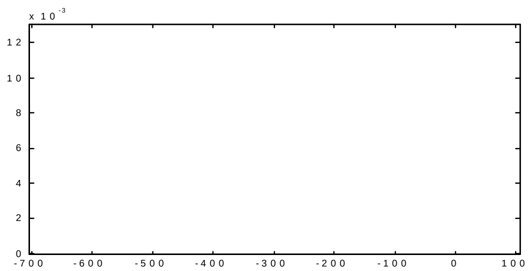

Figure 3b. 13 point Recursive Median filter, estimated sample selection probability function. Strongly biased dependency and the long tail are due to streaking.

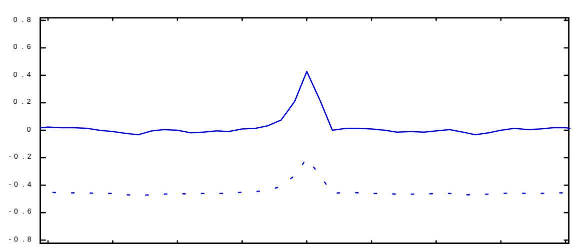

Figure 4a. A 2000 point i.i.d. normally distributed test signal filtered with the 3 point Recursive Median filter. Correlation with the input signal is computed with the method of L2-norm cross correlation (solid line), and L1-norm cross correlation (dotted line).

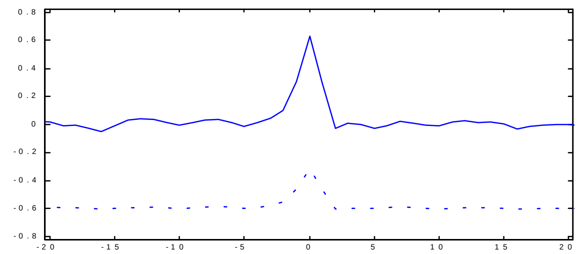

Figure 4.b. Same as above, except that the input signal is i.i.d. Laplacian.

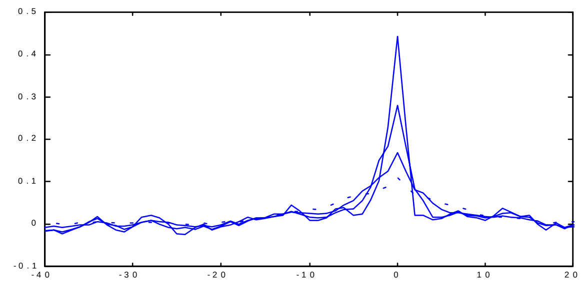

Figure 5a. Cascade structure. L2-correlation of scales 1, 2, 4 and 8 with the original, (scale 8 dotted line). Sharpest peaks belong to the lower scales.

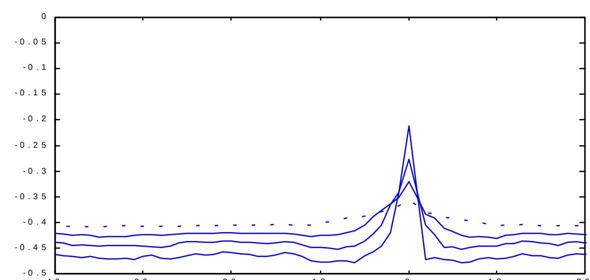

Figure 5b. Cascade structure. L1-correlation of scales 1, 2, 4 and 8 with the original, (scale 8 dotted line). The results are not normalised. General appearance is similar with figure 5a.

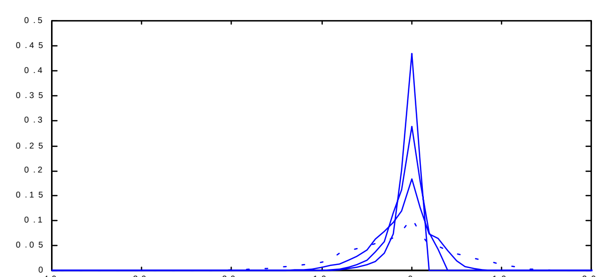

Figure 5c. Cascade structure. EQ-correlation of scales 1, 2, 4 and 8 with the original, (scale 8 dotted line). All 8 sample selection probability functions show easily detectable peaks, lowering steadily at higher scales.

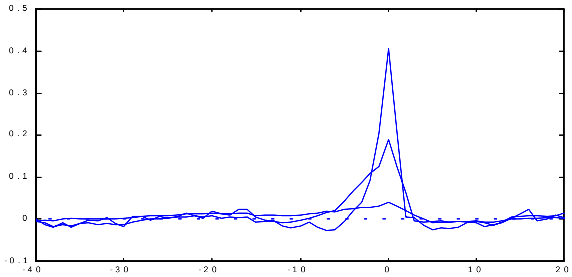

Figure 6a. Parallel structure. L2-correlation of scales 1, 2, 4 and 8, (scale 8 dotted line), with the original. At longer filter windows L2 correlation ceases to give meaningful results.

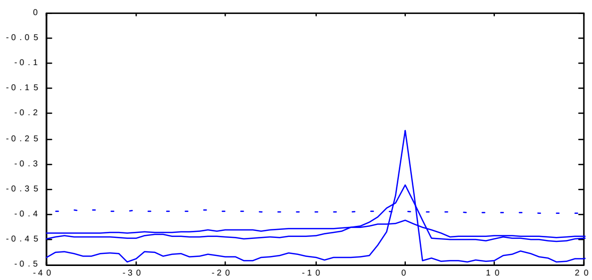

Figure 6b. Parallel structure. L1-correlation of scale 1, 2, 4 and 8, (scale 8 dotted line), with the original.

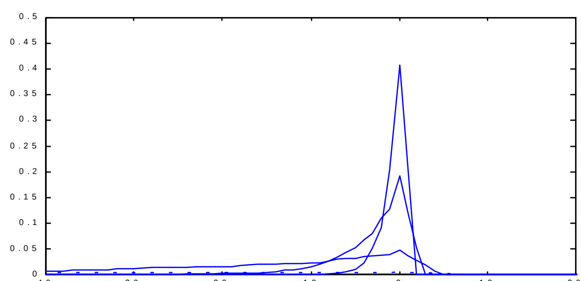

Figure 6c. Parallel structure. EQ-correlation of scales 1, 2, 4 and 8, (scale 8 dotted line), with the original. Sample selection probability functions show strong asymmetry at larger window lengths. This is due to massive streaking.

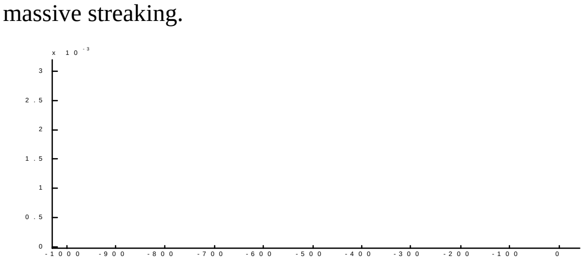

Figure 6d. Parallel structure. EQ-correlation of scale 8, (RM-filter with window length 17), with the original. Longest streaks are of the order of hundreds of samples.

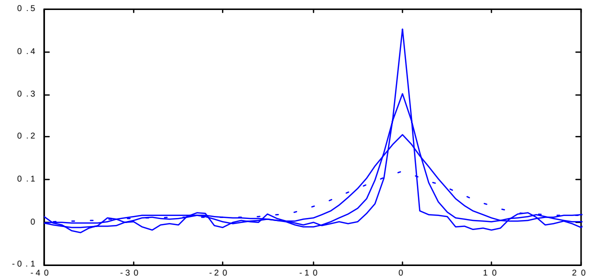

Figure 7a. Cascade structure. L2-correlation of scales 1, 2, 4 and 8, (scale 8 dotted line), with the previous scale.

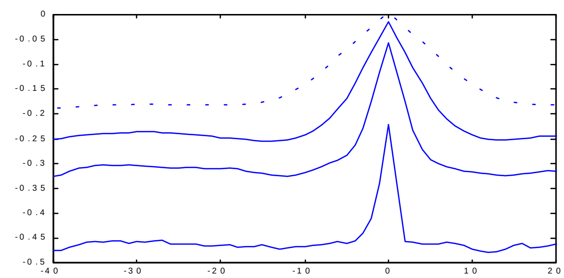

Figure 7b. Cascade structure. L1-correlation of scales 1, 2, 4 and 8, (scale 8 dotted line), with the previous scale.

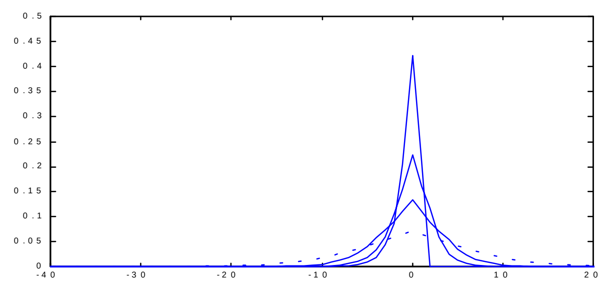

Figure 7c. Cascade structure. EQ-correlation of scales 1, 2, 4 and 8, (scale 8 dotted line), with the previous scale. Sample selection probability functions show near symmetry in all but the first scale, (the 3 point Recursive Median filter).