Fuzzy Stack Filtering Under Structural Constraints

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ABSTRACT

Rank-order based filters have received considerable attention due to their inherent outlier rejection and detail preservation properties. One of important rank-order based filters is stack filters. There are two different approaches to design of stack filters. One might be called the structural approach, while the other might be called the estimation approach. On the other hand, we have proposed fuzzy stack filter in order to extend the class of stack filters. Fuzzy stack filters are very important for signal processing, because these filters include FIR filters and weighted median filters. In this paper, we develop an approach for finding optimal fuzzy stack filters under the structural constraints. The noise attenuation of fuzzy stack filters is studied. Finally, we apply the proposed fuzzy stack filters to image restoration problem.

1. INTRODUCTION

Median type filters have the properties of impulse noise removal and highly signal preservation. In the median type filters, the widest class filter is stack filers [1], which include weighted median filers ,morphology filters , and so on. The stack filter is defined by Boolean function, then design method of the stack filter results in the design of Boolean function. There are two different approaches of designing stack filters. One might be called the structural approach[2],[3], while the other might be called the estimation approach[4],[5].

The stack filters are being studied with the filter defined by the positive Boolean function whose output restricted only two values (i.e., 0 or 1). We proposed fuzzy stack filters which are defined by fuzzy Boolean function[6],[7]. The output of fuzzy Boolean function takes value from 0 to 1 continuously. And we also proposed fuzzy median filters and fuzzy weighted median filters[6],[7],[8] which are special cases of fuzzy stack filters. In [6],[7],[8], it was shown that the fuzzy median and fuzzy weighted median filters included linear filters such as the mean filter and weighted average filters. Moreover, the design method of the estimation approach and the performance of these filters are also shown.

In this paper, we propose the fuzzy stack filter which can preserve some desired signal details (e.g., lines and corners in images). In [3], it was shown that that Boolean functions of the stack filters which preserve specific structures, could be obtained easily. These Boolean functions have two elements. One is the term which realizes preserving specific structures. Another is the term which is Boolean function of the median filter. Fuzzy stack filters which can preserve some desired signal details can be realized by substituting fuzzy median filter's Boolean function for latter term (i.e., Boolean function of the median filter). The noise attenuation properties of the proposed filter is studied. Furthermore, we show the effectiveness of the proposed filter for image restoration.

2. STACK FILTERING UNDER STRUCTUAL CONSTRAINTS

2.1. Stack Filters

Input signal $x(n)$ is assumed to be M-valued signal, the output of stack filters is given by way of the following three processes.

[The threshold decomposition]

$$x^m(n) = T^m(x(n)) = \begin{cases} 1: & \text{if } x(n) \geq m \\ 0: & \text{else} \end{cases} \quad (m = 1, \ldots, M-1) \tag{1}$$

$x(n)$ was decomposed into the set of M-1 binary signals $x^m(n)$.

[The calculating output of Boolean function]

$$y^m(n) = f(\mathbf{x}^m(n)) \quad (m = 1, \ldots, M-1) \tag{2}$$

where

$$\mathbf{x}^m(n) = [x^m(n-N), \ldots, x^m(n), \ldots, x^m(n+N)] \tag{3}$$

[The composing binary output signal]

Output of stack filters $y(n)$ is given by

$$y(n) = \sum_{m=1}^{M-1} y^m(n) \tag{4}$$

If we define Boolean functions appropriately, the stack filters are equivalent to the median filter, the weighted Median (WM) filter, and the morphological filter.

The median filter is the stack filter whose Boolean function is given by

$$f_{MED}(r^m(n)) = \begin{cases} 1: & \text{if } r^m(n) > 0.5 \\ 0: & \text{if } r^m(n) \le 0.5 \end{cases} \quad (5)$$

where

$$r^m(n) = \sum_{i=-N}^{N} x^m(n+i) \Big/ (2N+1) \quad (6)$$

2.2. Stack Filtering Under Structural Constraints

An image consists of many signal structures, such as lines and corners, that are critical to perception. It is desirable that stack filters can preserve such important image details and at the same time remove noise effectively.

Consider a stack filter with 3×3 square window

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{bmatrix}$$

The processed point is X_4. Three different sets of structural constrains are following, and the corresponding optimal stack filters are given.

A. Horizontal and vertical lines preservation (HV-S filter)

We can obtain the optimal noise attenuation stack filter which preserve horizontal and vertical lines as

$$f(x_0^m, \dots, x_8^m) = \{ f_{MED}(x_0^m, \dots, x_8^m) + x_3^m \cdot x_4^m \cdot x_5^m + x_1^m \cdot x_4^m \cdot x_7^m \} \cdot (x_3^m + x_4^m + x_5^m) \cdot (x_1^m + x_4^m + x_7^m) \quad (7)$$

If $x_3^m = x_4^m = x_5^m$ or $x_1^m = x_4^m = x_7^m$ is equal to 1, the value of $x_3^m \cdot x_4^m \cdot x_5^m$ or $x_1^m \cdot x_4^m \cdot x_7^m$ is equal to 1. Thus, output of Boolean functions is 1 regardless of the value of $f_{MED}(x_0^m, \dots, x_8^m)$. On the other hand , if $x_3^m = x_4^m = x_5^m$ or $x_1^m = x_4^m = x_7^m$ is equal to 0, the value of $x_3^m + x_4^m + x_5^m$ or $x_1^m + x_4^m + x_7^m$ is equal to 0. Then, the output of Boolean function is 0 regardless of the value of $f_{MED}(x_0^m, \dots, x_8^m)$. Now it is clear that the optimal stack filter is a composition of the median filter and a set of max and min filters. The max and min filters are determined by the structural constraints.

B. Horizontal, vertical and two diagonal lines preservation (HVD-S filter)

In the same manner as in HV-S filter, we obtain the optimal stack filter that preserves horizontal, vertical and two diagonal lines

$$\begin{aligned} f(x_0^m, \dots, x_8^m) = \{ & f_{MED}(x_0^m, \dots, x_8^m) + x_3^m \cdot x_4^m \cdot x_5^m + x_1^m \cdot x_4^m \cdot x_7^m \\ & + x_0^m \cdot x_4^m \cdot x_8^m + x_2^m \cdot x_4^m \cdot x_6^m \} \cdot (x_3^m + x_4^m + x_5^m) \\ & \cdot (x_1^m + x_4^m + x_7^m) \cdot (x_0^m + x_4^m + x_8^m) \cdot (x_2^m + x_4^m + x_6^m) \end{aligned} \quad (8)$$

C. Horizontal, vertical, two diagonal lines and four corners preservation (HVDC-S filter)

The optimal stack filter that preserves horizontal, vertical, two diagonal lines and four corners

$$\begin{aligned} f(x_0^m, \dots, x_8^m) = \{ & f_{MED}(x_0^m, \dots, x_8^m) + x_3^m \cdot x_4^m \cdot x_5^m + x_1^m \cdot x_4^m \cdot x_7^m \\ & + x_0^m \cdot x_4^m \cdot x_8^m + x_2^m \cdot x_4^m \cdot x_6^m + x_0^m \cdot x_1^m \cdot x_3^m \cdot x_4^m + x_2^m \cdot x_4^m \cdot x_6^m \\ & + x_0^m \cdot x_1^m \cdot x_3^m \cdot x_4^m + x_1^m \cdot x_2^m \cdot x_4^m \cdot x_5^m + x_3^m \cdot x_4^m \cdot x_6^m \cdot x_7^m \\ & + x_4^m \cdot x_5^m \cdot x_7^m \cdot x_8^m \} \cdot (x_3^m + x_4^m + x_5^m) \cdot (x_1^m + x_4^m + x_7^m) \\ & \cdot (x_0^m + x_4^m + x_8^m) \cdot (x_2^m + x_4^m + x_6^m) \cdot (x_0^m + x_1^m + x_3^m + x_4^m) \\ & \cdot (x_1^m + x_2^m + x_4^m + x_5^m) \cdot (x_3^m + x_4^m + x_6^m + x_7^m) \cdot (x_4^m + x_5^m + x_7^m + x_8^m) \end{aligned} \quad (9)$$

3. FUZZY STACK FILTERING UNDER STRUCTURAL CONSTRAINTS

3.1. Fuzzy Median Filters

The output of Boolean functions of the stack filters is restricted two values (i.e., 0 or 1). In [6],[7],[8], we attempt to enlarge class of stack filters by defining the output of the Boolean function from 0 to 1 continuously. We call this filter the fuzzy stack filter.

Fuzzy median filter[7] is the one of fundamental fuzzy stack filters, which is defined by the fuzzy Boolean function $f_{FM}(r^m(n))$ which is monotonous increase function. If the fuzzy Boolean function defined as $f_{FM}(r^m(n)) = r^m(n)$, fuzzy median filter is equivalent to the mean filter. In [7], we approximated the fuzzy Boolean function $f_{FM}(r^m(n))$ by the following sigmoidal function.

$$f_{FM}(r^m(n)) = 1 \big/ [1 + \exp\{-\alpha \cdot (r^m(n)) - \beta\}] \quad (10)$$

In equation (10), if $\beta = 0.5$ and α is large value, then, $f_{FM}(r^m(n)) = f_{MED}(r^m(n))$, thus, the fuzzy median filter becomes equivalent to the median filter. If value of α is less than 10, the fuzzy median filter becomes close to the mean filter. We can change the property of the fuzzy median filters by changing α.

3.2. Fuzzy Stack Filtering Under Structural constraints

From section 2.2, the Boolean function of stack filters which can preserve some specific structures, consists of the median filter and a set of max and min filters. The max and min filters are determined by structural constraints. Thus, we can obtain the fuzzy stack filters which preserve some desired structures by substituting fuzzy median filter for median filter in the Boolean function of stack filter which preserve some desired structures.

From this point of view, we define fuzzy Boolean functions of HV-fuzzy stack (HV-FS) filter, HVD-fuzzy stack (HVD-FS) filter and HVDC-fuzzy stack (HVDC-FS) filter, respectively, as follows.

A. HV-FS filter

$$f_F(x_0^m,\ldots,x_8^m) = \{f_{FM}(x_0^m,\ldots,x_8^m) + x_3^m \cdot x_4^m \cdot x_5^m + x_1^m \cdot x_4^m \cdot x_7^m\} \cdot (x_3^m + x_4^m + x_5^m) \cdot (x_1^m + x_4^m + x_7^m) \tag{11}$$

B. HVD-FS filter

$$\begin{aligned} f_F(x_0^m,\ldots,x_8^m) = \{ & f_{FM}(x_0^m,\ldots,x_8^m) + x_3^m \cdot x_4^m \cdot x_5^m + x_1^m \cdot x_4^m \cdot x_7^m \\ & + x_0^m \cdot x_4^m \cdot x_8^m + x_2^m \cdot x_4^m \cdot x_6^m\} \cdot (x_3^m + x_4^m + x_5^m) \\ & \cdot (x_1^m + x_4^m + x_7^m) \cdot (x_0^m + x_4^m + x_8^m) \cdot (x_2^m + x_4^m + x_6^m) \end{aligned} \tag{12}$$

C. HVDC-FS filter

$$\begin{aligned} f_F(x_0^m,\ldots,x_8^m) = \{ & f_{FM}(x_0^m,\ldots,x_8^m) + x_3^m \cdot x_4^m \cdot x_5^m + x_1^m \cdot x_4^m \cdot x_7^m \\ & + x_0^m \cdot x_4^m \cdot x_8^m + x_2^m \cdot x_4^m \cdot x_6^m + x_0^m \cdot x_1^m \cdot x_3^m \cdot x_4^m + x_2^m \cdot x_4^m \cdot x_6^m \\ & + x_0^m \cdot x_1^m \cdot x_3^m \cdot x_4^m + x_1^m \cdot x_2^m \cdot x_4^m \cdot x_5^m + x_3^m \cdot x_4^m \cdot x_6^m \cdot x_7^m \\ & + x_4^m \cdot x_5^m \cdot x_7^m \cdot x_8^m\} \cdot (x_3^m + x_4^m + x_5^m) \cdot (x_1^m + x_4^m + x_7^m) \\ & \cdot (x_0^m + x_4^m + x_8^m) \cdot (x_2^m + x_4^m + x_6^m) \cdot (x_0^m + x_1^m + x_3^m + x_4^m) \\ & \cdot (x_1^m + x_2^m + x_4^m + x_5^m) \cdot (x_3^m + x_4^m + x_6^m + x_7^m) \cdot (x_4^m + x_5^m + x_7^m + x_8^m) \end{aligned} \tag{13}$$

4. SIMULATION RESULTS

4.1. Noise Attenuation Property

We study the noise attenuation property of the proposed filters. We consider the input signal as (constant signal) + (noise: Gaussian, uniform and Laplacian distributions with variance 1) (we use 256x256 samples in this simulation). In Table 1 results for the filters output variance are given in the case where 3x3 window which show the noise attenuation ability of each filter. Fuzzy stack filters under structural constraints are superior to stack filter under structural constraints in noise attenuation property, especially, the noise distribution tail length is relative short.

Table 1. Noise Attenuation Property

Filter	Uniform	Gaussian	Laplacian
Mean	0.112	0.112	0.112
Median	0.279	0.156	0.068
HV-S	0.309	0.187	0.089
HV-FS	0.271	0.176	0.089
HVD-S	0.345	0.217	0.106
HVD-FS	0.333	0.216	0.106
HVDC-S	0.353	0.222	0.109
HVDC-FS	0.345	0.222	0.109

4.2. Application to Image Restoration

In this section we present some experiment results. All filter's windows size set 3x3. We prepare 5 filters (HVD-S, HVD-FS, HVDC-S, HVDC-FS and HVDC-Weighted median (HVDC-WM) filters) for image restoration. Parameter α (in equation (10)) of HVD-FS and HVDC-FS filters fixed 6 for all input images. This value is determined by a lot of experimental results. The HVDC-WM filter is the WM filter which preserve the horizontal and vertical lines, two diagonal lines and four corners[9]. This WM filter's weights is given by

$$\begin{bmatrix} w_0 & w_1 & w_2 \\ w_3 & w_4 & w_5 \\ w_6 & w_7 & w_8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{14}$$

Original images "Lena" and "Boat" are corrupted by various distribution noises (uniform, Gaussian and Laplacian distribution). Each noise variance is 100 and 400. Table 2 shows the restoration results.

Compared to the stack and fuzzy stack filters listed in Table 2, we can observe that fuzzy stack filters give the more details are to be preserved, the more noise reduction is achieved. In the case of high signal to noise ratio input images, the HVDC-FS filter is superior to the HVC-FS filter. On the other hand, if input images are low signal to noise ratio, the HVC-FS filter is superior to HVDC-FS filter.

Anyway, fuzzy stack filters which preserve desired structures proposed in this paper, work well for image restoration problem.

5. CONCLUSION

In this paper, we show optimal fuzzy stack filters under the structural constraints. The noise attenuation property of the proposed fuzzy stack filters is analyzed experimentally. From this analysis, fuzzy stack filters under the structural constraints are superior to stack filters under the same structural constraints, especially, the tail length of noise distribution is short. Furthermore, we apply the fuzzy stack filter under structural constraints to image restoration. The proposed fuzzy stack filters work well for image processing.

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Table 2 Results of Filtering
(Normalized Mean Square Error: NMSE)

Image	(*)	variance	HVD-S	HVD-FS	HVDC-S	HVDC-FS	HVDC-WM
Lena	U	100	0.812	0.665	0.772	**0.625**	0.734
	G		0.745	0.605	0.707	**0.506**	0.628
	L		0.597	0.480	0.554	**0.433**	0.460
	U	400	0.489	**0.419**	0.461	0.435	0.623
	G		0.373	**0.345**	0.371	0.354	0.495
	L		0.255	**0.241**	0.250	0.243	0.331
Boat	U	100	0.680	0.566	0.646	**0.522**	0.699
	G		0.603	0.500	0.562	**0.478**	0.588
	L		0.462	0.389	0.426	**0.359**	0.432
	U	400	0.441	**0.404**	0.444	0.428	0.627
	G		0.341	**0.322**	0.339	0.337	0.494
	L		0.231	**0.225**	0.227	0.232	0.329

* U : uniform distribution, G: Gaussian distribution, L: Laplacian distribution