APPLICATIONS OF THE FRACTIONAL FOURIER TRANSFORM TO CORRELATION, FEATURE EXTRACTION AND PATTERN RECOGNITION

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ABSTRACT

This paper presents a brief overview of various signal processing applications based on the fractional Fourier transfromation and its related, recently-defined, phase spaces.

1. INTRODUCTION

One of the first practical optical approaches for performing correlation is the well known Vander Lugt 4-f coherent configuration [1], its analogous incoherent system [2] or using the joint transform correlator [3, 4]. Since the conventional correlation is a shift invariant operation, shifts of the input pattern provide a shifted correlation output plane with no effect on the field distribution, and pixels located close to the center has exactly the same effect as pixels located at the outer area.

In several pattern recognition applications the shift invariance property within all of the input plane is not necessary and even disturbs . An example is the case where the object is to be recognized only when its location is inside a certain area and rejected otherwise, e.g. a label which has been affixed in the incorrect place during manufacturing. Several approaches for obtaining such space variance detection have been suggested. The first approach is the Tandem component processor that trades the shift invariance with high efficiency and high peak to correlation- energy ratio [5]. Different approach is based on coded phase processor which multiplexed many filters and yet kept the space bandwidth product (SW) of the ordinary single filter correlator [6]. Recently, a space-variant Fresnel transform correlator [7], which is closely related to a lensless intensity correlator [8], was suggested.

A similar approach is the tool coined fractional correlation (FC) whose optical implementation is done using a setup similar to the Vander Lugt correlator [9, 10]. Opposed to a solution of using an appropriate input pupil which is open in the desired location, the FC does not require any additional equipment for its optical implementation. The FC itself selects the area of interest within the input scene. Additional example for the necessity of the FC is the case where the recognition should mainly be based on the central pixels and less on the outer pixels (for instance in systems whose spatial resolution is improved in the central pixels, and thus the central region of pixels is more reliable for the recognition process). An important application for the FC might be the detection of localized objects using a single cell detector, eliminating the need for a CCD array detector.

The FC is a generalization of the conventional correlation operation and it is based on the fractional Fourier transform (FRT) [11]. The FRT operation is useful for various spatial filtering and signal processing applications [12, 13], that is defined through a transformation kernel, as illustrated in Ref. [13]:

$$\{\mathcal{F}^p u(x)\}(x) = \int_{-\infty}^{\infty} B_p(x, x')u(x')dx' \qquad (1)$$

where $B_p(x, x')$ is the kernel of the transformation and p is the fractional order.

$$B_p(x, x') = \frac{\exp\left[-i\left(\frac{\pi\, sgn(\sin\phi)}{4} - \frac{\phi}{2}\right)\right]}{|\sin\phi|^{\frac{1}{2}}} \cdot \exp\left[i\pi\frac{x^2 + x'^2}{\tan\phi} - 2i\pi\frac{xx'}{\sin\phi}\right]$$
$$\phi = \frac{p\pi}{2} \qquad (2)$$

This kernel has two optical interpretations, one as a propagation through GRIN medium [11] and the second as a rotation operation applied over the Wigner plane [14]. Both definitions were shown to be fully equivalent in Ref. [15].

2. FRACTIONAL CORRELATION

The fractional correlation (FC) operation allows to control the amount of shift variant property of the correlation. This property is based on the shift variance of the FRT and it is more significant for the fractional orders of $p \approx 0 + 2N$ and less for $p \approx 1 + 2N$ (N is any integer).

It consists of obtaining the product of the fractional transforms of the distributions to be correlated, rendering a last FRT to obtain the final result. Analytically, the operation of FC of an input function, $f(x)$, with a reference pattern, $g(x)$, is defined as follows:

$$C_{p_1,p_2,p_3}(x') = \mathcal{F}^{p_3}\left\{\mathcal{F}^{p_1}\left\{f(x)\right\}\mathcal{F}^{p_2}\left\{g(x)\right\}\right\} \qquad (3)$$

Where p_1, p_2, p_3 are the orders of the FRTs to perform, in principle arbitrary. Due to various reasons, detailed in Ref. [9], the most obvious choice is:

$$p_1 = p \qquad p_2 = -p \qquad p_3 = -1 \qquad (4)$$

with p ranging from 0 to 1. In this case, if the input coincides with the reference object, a perfect phase matching between object and reference FRTs in the fractional domain is obtained. The inverse Fourier transform will just focus the resulting plane wave.

In order to build optically a FC, instead of preparing a full setup containing two lenses and free propagations, the object is illuminated with a converging beam [16].

This permits the change of the convergence phase factor, multiplying the object, by displacing it along the optical axis. The matching between the distance object-filter and the convergence of the beam may produce any desired order and scaling factor. Hence, this approach is more convenient for the experimenter, as the exact sizes of the input and filter transparencies are often not precisely determined. This is especially important for the case of using SLMs for implementing the filter. As the FRT is not exact there will be a quadratic phase factor multiplying the output plane. It means that the correlation plane will be displaced along the optical axis.

3. APPLICATIONS

Besides the ability to change easily the space variance of the optical system, the FRT has shown to be very useful for many other applications in signal processing. The main application is related to chirp noise removal. This application is based on the fact that if a chirp type noise of $\exp(-i\alpha x^2)$ is fractionally Fourier transformed with the order of $p = \frac{2}{\pi}\tan^{-1}\left(\frac{1}{\alpha}\right)$ the result is essentially a delta function. Thus, in order to remove the noise a simple notch filter may be placed in the proper FRT plane, using the fractional correlator configuration. Since the filter is notch, the amount of the signal's information lost, is minimal. Another important application of the FRT is related to the fact that the FRT corresponds to rotation of the Wigner chart by an angle of $p\frac{\pi}{2}$. Thus, assuming that the Wigner chart for the signal and the noise distributions is as illustrated by Fig. 1, one may see that filtering either in the Fourier plane (corresponds to projection of the Wigner chart over the f_x axis) or filtering in the spatial plane (corresponds to projection of the Wigner chart over the x axis) will result in partial lose of the signal's information. However, filtering in the proper FRT plane (the angle in the Wigner chart at which full separation exists between the projections of the signal and the noise) may result in full reconstruction of the signal out of its noise [17].

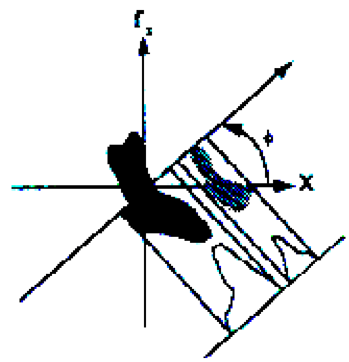
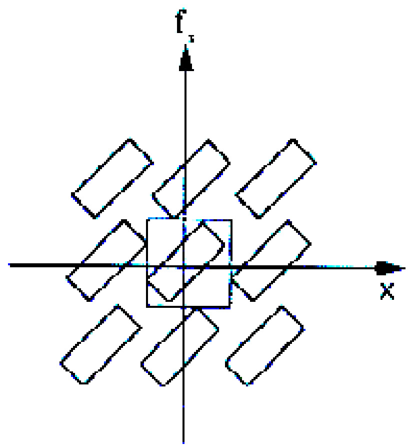


Figure 1: A Wigner chart of a signal and a noise where the FRT filtering is very applicable.

Another important application of the FRT is related to signal multiplexing. Due to the ability of the FRT to rotate the Wigner chart, the Wigner distribution of a signal may be arranged in a more efficient manner [18] as seen in Fig. 2. That efficient arrangement saves additional bandwidth that may be needed for the transmission of the signal. Note that Fig. 2 is based on the fact that shifting a signal in space, shifts its Wigner chart in the spatial axis. Multiplying the signal by a linear phase factor shifts its Wigner chart in the frequency axis accordingly to the linear phase.

4. THE ADAPTIVE FRACTIONAL PROCESSOR

The FC discussed so far was based on the FRT with an uniform fractional order applied over the reference and the input functions. In this section we introduce an adaptive FRT (AFRT), i.e an FRT whose fractional order is space dependent and thus the amount of shift variance/invariance is also spatially controlled. Such a transformation may be implemented optically in FC configurations achieving both: shift variant noise removal (for non stationary noises whose statistical prop-



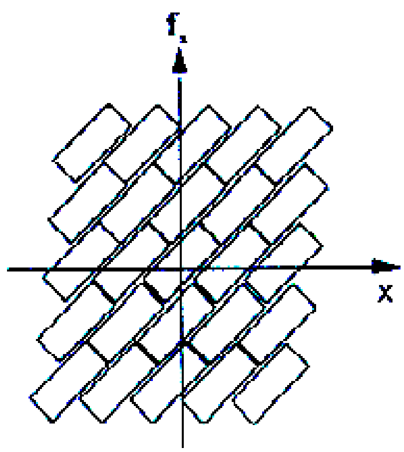


Figure 2. A Wigner chart of a signal which may be more efficiently multiplexed via an FRT.

erties are varied with the spatial position) and image detection (for many detection applications which could be implemented with better efficiency by using a shift variant or partially shift variant processors).

The most common case where different spatial shift variant processing is required, relates to finger print recognition [18]. The finger print is a pattern whose spatial variance is changed with the spatial location. Its central region is more or less constant while the outer one is changed from instant to instant since one never presses his finger in equal force. Thus, in order to recognize or reconstruct those prints, a processor whose spatial shift variance is changed, is required. Due to the physical construction of the filter, in the center a small shift invariance is needed, but in the outer regions of the print increasing shift invariance is required. In this practical case an AFRT processor should be helpful, since an efficient recognition of the finger prints can be very applicable, for instance, in safety lockers or in entrance to permission restricted entrance zones.

5. THE (X, P) CHART

Recently in the digital processing and the computerized tomography field, a new time-frequency analyzing tool, the Radon Wigner transform, was suggested [19, 20] and used for time frequency-representation of digital signals [21, 22]. This approach led to a chart that contains a continuous representation of the FRT of a signal as a function of the FRT order [23]. Concerning us, this representation may be useful also in optics since it shows explicitly the propagation of a signal inside a graded index medium. The given approach for producing this display starts with a 1-D input signal while the output signal contains 2 D. The optical setup for obtaining the FRT was adapted to include only fixed free space propagation distances and variable lenses. With a set of two multi facet composite holograms, the Radon-Wigner display has been experimentally demonstrated.

5.1. Mathematical Definition

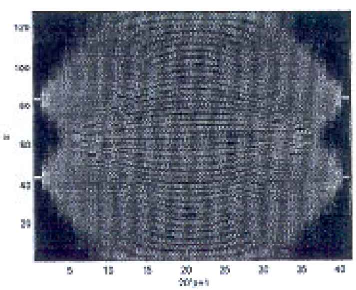
A display that contains a continuous representation of an FRT of a signal as a function of the FRT order is coined by us the (x, p) display, and may be useful both for digital signal processing (see Ref [21]) and for optics. For a 1-D object, this plot contains two axes: the vertical is the space coordinate x and the horizontal is the FRT order p. The 1-D light distribution $u_p(x)$ (a p order FRT of the original signal $u_0(x)$) is placed as a strip in the proper horizontal location in the chart according to its fractional order p. More explicitly one can write

$$F(x,p) = u_p(x) \quad (5)$$

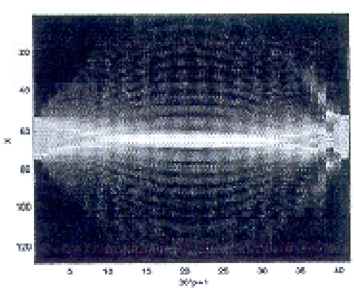
As a result, all of the FRT orders of the original function $u_0(x)$ are calculated and displayed in one plot.

5.2. Simple Examples

Fig. 3 illustrates the (x,p) charts of a two simple signals. In Fig. 3a the illustrated signal is $u_1(x) = \delta(x - x_0) + \delta(x + x_0)$ and in 3b the signal is $u_2(x) = rect\left(\frac{x}{\Delta x}\right)$.



a).



b).

Figure 3: An example of the (x,p) display. a). $u_1(x) = \delta(x - x_0) + \delta(x + x_0)$ b). $u_2(x) = rect\left(\frac{x}{\Delta x}\right)$.

One may see, in both figures, how the expected Fourier transform is obtained in $(x, p = 1)$ (for u_1 a cosine and for u_2 a sinc function).

6. THE (R,P) CHART

The next step after defining the (x,p) chart is what we call an (r,p) chart. This chart performs a Cartesian to polar coordinate transform of the (x,p) chart [24]. Here, all of the FRT orders of the function are drawn as angular vectors. Each FRT order is drawn along the r axis in specific angular orientation of $\phi = p\frac{\pi}{2}$ where p is the fractional order. Implicitly, one can write the (r,p) representation as

$$F(r,p) = u_p(r) \quad (6)$$

Fig. 4 gives a graphical illustration of the (r,p) chart representation.

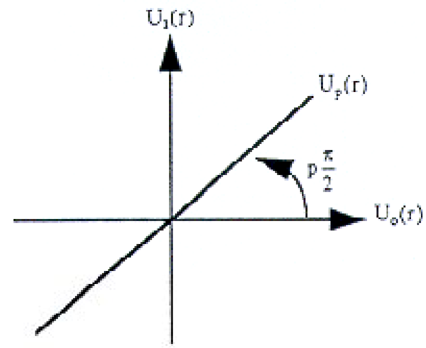


Figure 4: Illustration of the (r,p) chart.

It is important to note that despite r being a radial coordinate it may get negative values. The r coordinate negative values are a by product of the (r,p) chart definition. However, speaking about negative values for r has no conflict with the polar coordinate definition since

$$u_{p+2}(r) = u_p(-r) \quad (7)$$

Another note is connected with $r = 0$. This singular point contains no relevant information and should be avoided while using the chart. As a polar representation, the required spatial resolution for lower r value is higher. Thus, practically, a certain area of $|r| < r_0$ is not able to carry the necessary information (due to the limited spatial resolution of every plot) and must be avoided too.

The (r,p) chart is our candidate for serving as a phase space representation. It contains full information about the object (along $\phi = 0$) and about its spectrum (along $\phi = \frac{\pi}{2}$). Additional information regarding the mixture space-frequency information is given along other values of ϕ. The inverse transformation is trivial

$$u_p(r) = F(r,p) \quad (8)$$

and for the object itself

$$u_0(r) = F(r,0) \quad (9)$$

7. THE USAGE OF THE (X,P) AND THE (R,P) CHARTS FOR SIGNAL PROCESSING

In this section we will illustrate the usage of the (x,p) and the (r,p) charts for identification of acoustic signals.

The Wavelet transform is a very common tool for speech identification [25, 26]. However, in order to obtain high discrimination between similar acoustic signals, proper algorithms for choosing the optimal decomposition base are to be applied. An improved discrimination may also be obtained by using the (x,p) or the (r,p) displays. In order to perform the identification the (x,p) or the (r,p) charts of the reference pattern are computed and stored in the computer memory. The input sounds sequence is divided into patterns having similar lengths and their (x,p) or (r,p) charts are calculated. A 2-D correlation is performed between the (x,p) or (r,p) charts of the reference and the input sound patterns:

$$\begin{aligned} C_{xp}(x,p) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{xp}(x',p') \cdot \\ &\cdot \; F^{*}_{xp}(x'-x,p'-p)dx'dp' \\ C_{rp}(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{rp}(x',y') \cdot \\ &\cdot \; F^{*}_{rp}(x'-x,y'-y)dx'dy' \qquad (10) \end{aligned}$$

where G_{xp} and F_{xp} are the input and the reference (x,p) charts, respectively. $G_{rp}(x,y)$ and $F_{rp}(x,p)$ are the input and the reference (r,p) charts, respectively, where the correlation is performed in Cartesian coordinates set x and y.

Apparently, such an algorithm obtains supreme discrimination ability resulted in the additional dimension of the (x,p) or the (r,p) charts. Fig. 5 illustrates the processing abilities of the (x,p) chart on an acoustic signal. The input acoustic signal is shown in Fig. 5a. The acoustic pattern was sampled in a sampling rate of 8192 Hz. A pattern of 128 pixels (15.63 [msec]) taken from the middle of the input signal, was chosen to be the reference signal. Fig. 5b presents the (x,p) chart of the reference signal. Fig. 5c presents the cross section of the obtained correlation as function of time. For comparison, a cross section of the correlation between the reference signal and a similar acoustic signal (seen in Fig. 5d), is presented in Fig. 5e. One may see that much lower correlation values are obtained.

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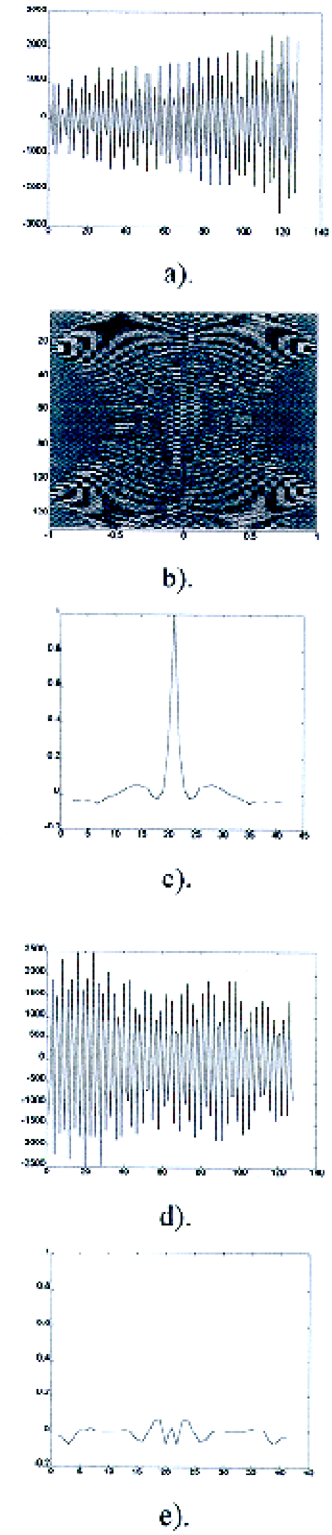

Figure 5: An example of the processing abilities of the (x,p) chart on acoustic signals. a). The input signal sequence. b). The (x,p) chart of the reference signal. c). The cross section of the obtained correlation. d). Additional input signal. e). The cross section of the obtained correlation.

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