PROPERTIES OF EIGENFUNCTIONS OF THE CANONICAL INTEGRAL TRANSFORM

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ABSTRACT

The structure and the properties of the eigenfunctions of the canonical integral transform are investigated. It is shown that a signal can be decomposed into a set of the orthogonal eigenfunctions of the generalized Fresnel transform. The property that the set contains a finite number of functions is obtained.

The canonical integral transform, also known as the generalized Fresnel transform (GFT) [1, 2], including as a particular case the fractional Fourier transform, is now actively used in optics, quantum theory, signal and image processing, etc. The GFT of a signal $f(x)$ is given by

$$F_M(u) = R^M\left[f(x)\right](u) = \int_{-\infty}^{\infty} f(x) K_M(x,u)\,dx, \quad (1)$$

with the kernel

$$K_M(x,u) = \begin{cases} \dfrac{1}{\sqrt{iB}} \exp\left(i\pi \dfrac{Ax^2 + Du^2 - 2xu}{B}\right), B \neq 0 \\ \dfrac{1}{\sqrt{A}} \exp\left(i\pi \dfrac{Cu^2}{A}\right) \delta\left(u - x/A\right), B = 0 \end{cases} \quad (2)$$

parametrized by a real 2×2 matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (3)$$

with the determinant equal to 1.

The GFT parametrized by the matrix $A = D = \cos\alpha$ and $B = -C = \sin\alpha$ corresponds, except for a factor $\exp(i\alpha/2)$, to the fractional Fourier transform (FT) [3]. The eigenfunctions of the fractional Fourier transform have been investigated in Refs. [4, 5, 6, 7].

In this paper we consider the structure and the properties of the eigenfunctions for the GFTs and propose a method for their generation. We show that any signal can be decomposed into a set of orthogonal eigenfunctions of the GFT. This set contains a finite number of functions k if

$$\arccos((A + D)/2) = 2\pi m/k,$$

where k and m are integers.

A signal $f(x)$ is an eigenfunction $f_M(x)$ of the canonical operator R^M (a so-called self-GFT function) if

$$R^M\left[f_M(x)\right](u) = a f_M(u), \quad (4)$$

where $a = \exp(i2\pi\varphi)$ is a complex constant factor [1]. As it follows from Parseval's relation for the canonical transform of a signal with finite energy $\int |f(x)|^2\,dx < \infty$: $|a| = 1$ and therefore φ is real. Note that for infinite signals φ can be complex.

It has been shown in Ref. [1] that the functions

$$\Phi_n(x) = \left(\sqrt{\pi}2^n \lambda n!\right)^{-1/2} \exp\left(-\frac{(1+i\beta)}{2\lambda^2}x^2\right) H_n(x/\lambda) \quad (5)$$

are eigenmodes for the GFT operator R^M with eigenvalue $a = \exp(-i(n+1/2)\theta)$, where $H_n(u)$ are the Hermite polynomials, and where the parameters θ, λ, and β are defined from the parameters of the transfer matrix as

$$\begin{aligned} \theta &= \arccos\left((A+D)/2\right) \\ \lambda^2 &= 2B\left(4 - (A+D)^2\right)^{-1/2} \\ \beta &= (A-D)\left(4 - (A+D)^2\right)^{-1/2}, \end{aligned} \quad (6)$$

with $|A + D| \neq 2$ and $B, C \neq 0$. One can also write the expressions for the parameters of the transfer matrix as

$$\begin{aligned} A &= \cos\theta + \beta\sin\theta \\ B &= \lambda^2 \sin\theta \\ C &= -(\beta^2+1)\lambda^{-2}\sin\theta \\ D &= \cos\theta - \beta\sin\theta. \end{aligned} \quad (7)$$

The application of the relationships (5) and (7) for the construction of the eigenfunctions in limiting cases like the Fresnel transform ($\lambda^2 \to \infty$ and $\theta \to 0$) and the scaling transform ($\lambda^4 \to 0$ and $\beta^2 + 1 \to 0$) is problematic. We therefore confine ourselves to the GFT parametrized by a matrix for which $|A + D| \neq 2$ and $B, C \neq 0$. Note that β is equal to 0 only for the case $A = D = \cos\theta$. Then we have $B = \lambda^2 \sin\theta$ and $C = -\sin\theta/\lambda^2$, which represents the scaled fractional Fourier transform for real θ.

It is well known that the GFT operator R^M produces a linear transformation of the Wigner distribution $W_f(x,\omega)$ of the signal $f(x)$ in the phase space:

$$W_{R^M[f]}(x,\omega) = W_f(Dx - B\omega, A\omega - Cx).$$

Then the Wigner distribution of a self-GFT function is invariant under an affine transformation:

$$W_{f_M}(x,\omega) = W_{f_M}(Dx - B\omega, A\omega - Cx).$$

It follows from the cascading property for the GFT: $R^{M_2}R^{M_1} = R^{M_3}$, where $M_3 = M_2 \times M_1$, and Eq. (4), that the eigenfunction $f_M(x)$ for the canonical integral operator R^M with eigenvalue a, is also an eigenfunction with eigenvalue a^l for the GFT parametrized by the matrix M^l, where l is an integer. Then the function $\Phi_n(x)$ defined by Eq. (5) is also an eigenfunction for the GFT parametrized by the matrix M^l, whose parameters can be written as

$$\begin{aligned} A^{(l)} &= \cos l\theta + \beta \sin l\theta \\ B^{(l)} &= \lambda^2 \sin l\theta \\ C^{(l)} &= -(\beta^2+1)\lambda^{-2} \sin l\theta \\ D^{(l)} &= \cos l\theta - \beta \sin l\theta . \end{aligned} \quad (8)$$

From the linearity of the GFT and from the definition (4) it follows that a sum of eigenfunctions for a given GFT operator R^M with identical eigenvalues a is also an eigenfunction for R^M with the same eigenvalue a. Then a self-GFT function with eigenvalue $a = \exp(-i2\pi\varphi)$ can be represented as a superposition of certain modes $\Phi_n(x)$ with the indices $\{n\}$ satisfying the relationship

$$2\pi(N+\varphi) = -(n+1/2)\theta, \quad (9)$$

where φ is a constant defining the eigenvalue of this eigenfunction and N is an integer. Let n_1 and n_2 be solutions of this equation. Then we obtain that

$$2\pi(N_1 - N_2) = -(n_1 - n_2)\theta.$$

It is easy to see from this relationship that for a matrix for which the parameters A and D are such that $\theta/2\pi = \arccos\left((A+D)/2\right)/2\pi$ is complex or irrational, $n_1 = n_2$ is the only solution of Eq. (9). Therefore the functions $\Phi_n(x)$ are the only solutions of Eq. (4).

It follows from Eq. (6) that if $|A+D| > 2$, the parameters θ, λ^2, and β become complex:

$$\begin{aligned} \theta &= \pi k + i(-1)^k \text{arccosh}\left((A+D)/2\right) \\ \lambda^2 &= -i2B\left|4-(A+D)^2\right|^{-1/2} \\ \beta &= -i(A-D)\left|4-(A+D)^2\right|^{-1/2} . \end{aligned}$$

As an example let us consider the eigenfunctions for the GFT parametrized by the matrix

$$M = \begin{pmatrix} \cosh\alpha & \eta^2 \sinh\alpha \\ \sinh\alpha/\eta^2 & \cosh\alpha \end{pmatrix} \quad (10)$$

with real α and η. Since $A = D$, it follows from Eq. (7) that $\beta = 0$ and $B = -\eta^4 C$, which yields $\lambda^2 = i\eta^2$ and $\theta = i\alpha$. The set of orthonormal eigenmodes (5) with eigenvalues $a = \exp((n+1/2)\alpha)$ for this system can now be written as

$$\begin{aligned} \Phi_n(x) &= \left(\sqrt{\pi}2^n \exp(i\pi/4) n!\right)^{-1/2} \\ &\quad \times \exp\left(\frac{ix^2}{2\eta^2}\right) H_n\left(\exp(-i\pi/4)\frac{x}{\eta}\right). \end{aligned} \quad (11)$$

Thus the chirp function

$$\Phi_0(x) = \left(\sqrt{\pi}\exp(i\pi/4)\right)^{-1/2} \exp\left(\frac{ix^2}{2\eta^2}\right)$$

is self-reproducible under the GFT parametrized by the matrix (10). Note that the eigenvalues of the different modes $\Phi_n(x)$ and $\Phi_m(x)$ for the same angle α are different. This means that a superposition of these modes is not an eigenfunction of the corresponding GFT.

For the case $|A+D| < 2$, the parameters θ, λ^2, and β are real. Moreover we can always take $\lambda^2 > 0$ in Eq. (6) by choosing the appropriate sign of $\theta = \pm\arccos\left((A+D)/2\right)$.

If the parameters of the transfer matrix are such that $\theta/2\pi = \pm\arccos\left((A+D)/2\right)/2\pi$ is rational, $\theta = 2\pi m/k$ where k and m are relatively prime integers and $m < k$, then there are several sets of indices $\{n\}$ which satisfy Eq. (9). Since the eigenfunction for the GFT parametrized by a matrix M for which $\theta = 2\pi m/k$, is also an eigenfunction for the cascade of such GFTs with parameters defined by Eq. (8), it is more easy to construct the eigenfunction related with the matrix M^l such that $\theta l = 2\pi ml/k = 2\pi N + 2\pi/k$, where N is an integer. Using the periodicity property of trigonometric functions we obtain that the eigenfunction for the GFT parametrized by a matrix M with $\theta = 2\pi m/k$, is the eigenfunction for the GFT related to the matrix $M^{1/m}$ with $\theta = 2\pi/k$. It is easy to see that for $\theta = 2\pi/k$ we have k different sets of modes for which relation (9) holds: $n = L + kl$, where L is an integer constant from $[0,k[$ and l is an integer. Then a self-GFT function with eigenvalue $a = \exp(-i2\pi(L+1/2)/k)$ is defined as

$$f_k^L(u) = \sum_{l=0}^{\infty} g_{L+kl}\Phi_{L+kl}(u), \quad (12)$$

where g_{L+kl} are complex constants. This function is also an eigenfunction with eigenvalue $a = \exp(-i2\pi(L+1/2)m/k)$ for the GFT parametrized by the matrix M^m for any integer m.

If k is even, then as it follows from Eqs. (12) and (5), the self-GFT function is even for even L: $f_k^L(-u) = f_k^L(u)$, and odd for odd L: $f_k^L(-u) = -f_k^L(u)$.

It is easy to see that if $f_M(x)$ is an eigenfunction for the operator R^M with eigenvalue a, then its complex conjugate $f_M^*(x)$ is also an eigenfunction for $R^{\widetilde{M}}$, where $\widetilde{A} =$

$D, \widetilde{D} = A$, and $\widetilde{B} = B$, with the same eigenvalue a. Indeed if $|A + D| < 2$, then the complex conjugate of the eigenmode $\Phi_n^*(x)$ can be derived from $\Phi_n(x)$ by changing in Eq. (5) β to $-\beta$, which corresponds, in accordance with Eq. (6), to the matrix $\widetilde{M}$. For the case of the fractional FT or scaled fractional FT ($\beta = 0$) we obtain that if $f_\alpha(x)$ is an eigenfunction for the fractional FT operator R^α with eigenvalue a then its complex conjugate $f_\alpha^*(x)$ is also an eigenfunction for the same operator R^α with the same eigenvalue.

As well as in the case of the self-fractional Fourier functions, the self-GFT functions for the same operator with different eigenvalues (i.e., different indices L) are orthogonal to each other, because they are expanded into disjoint series of the orthogonal functions Φ_n.

Due to the fact that the functions Φ_n form a complete orthogonal set, any function $g(u)$ can be represented as their superposition

$$g(u) = \sum_{n=0}^{\infty} g_n \Phi_n(u). \tag{13}$$

Subdividing the series into partial ones

$$g(u) = \sum_{L=0}^{k-1} \left(\sum_{l=0}^{\infty} g_{L+kl} \Phi_{L+kl}(u) \right) = \sum_{L=0}^{k-1} f_k^L(u) \tag{14}$$

we have that a function $g(u)$ can be represented as a linear superposition of k orthogonal self-GFT functions $f_k^L(u)$ of a given operator R^M, where $\theta = 2\pi/k$. For $k = 2$ we obtain the function decomposition into the even and odd parts. Note that there are a number of operators described by the same θ, which differ from each other by the parameters λ and β.

On the other hand a self-GFT function for the operator R^M parametrized by a matrix M such that $\theta = 2\pi m/k$, can be constructed from any generator function $g(x)$ through the following procedure

$$f_k^L(u) = C \sum_{p=0}^{k-1} \exp\left(\frac{i2\pi(L + \frac{1}{2})p}{k} \right) R^{M^l} \left[g(x) \right](u), \tag{15}$$

where C is an arbitrary complex constant. This can be proved by using Eqs. (13) and (15). Indeed,

$$\begin{aligned} &\sum_{p=0}^{k-1} \exp\left(\frac{i2\pi(L + \frac{1}{2})p}{k} \right) R^{M^l} \left[\sum_{n=0}^{\infty} g_n \Phi_n(u) \right](x) \\ = \quad &\sum_{n=0}^{\infty} \sum_{p=0}^{k-1} \exp\left(\frac{i2\pi((L-n)p - (n+1/2))}{k} \right) g_n \Phi_n(x) \\ = \quad &k \sum_{n=0}^{\infty} \delta_{n,L+kl} \exp\left(-\frac{i2\pi(n+1/2)}{k} \right) g_n \Phi_n(x) \\ = \quad &k \exp\left(-i2\pi(L+1/2)\right) \sum_{l=0}^{\infty} g_{L+kl} \Phi_{L+kl}(x). \end{aligned}$$

Then choosing the constant C in Eq. (15) according to $C = k^{-1} \exp\left(i2\pi(L + 1/2)\right)$, we obtain that this procedure repeated for $L = 0, \ldots, k-1$ corresponds to the signal decomposition into a set of k orthogonal self-GFT functions $f_k^L(u)$ of a given operator R^M, where $\theta = 2\pi/k$.

We finally note that the signal decomposition on the finite set of the orthogonal GFT-functions for given R^M can be useful for signal analysis, filtering and securing information.

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