ROBUST SPECTRAL ESTIMATION BASED ON ARMA MODEL EXCITED BY A t-DISTRIBUTION PROCESS

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ABSTRACT

This paper will focus on proposing a new objective function of autoregressive moving average (ARMA)-model-based spectral estimation. The objective function is derived by assuming that the obtained residual signal is identically and independently distributed. The probability density function is assumed to be t-distribution. Small portions of residual signals with large amplitude is given a small weighting factor and large portions of residual signals with small amplitude is assiggned a large weighting factor. By doing so, the effect of large amplitude error signal is suppressed and the adaptation step is less affected. The simulation results for image enhancement show that when the input is impulsive noise contaminated images, the obtained processed image by using t-distribution with small α degrees of freedom is much better than that when large α is applied.

1. INTRODUCTION

Due to its simplicity, autoregressive (AR)-model-based spectral estimation which contain only zeroes has been commonly used in parametric spectral estimation, such as in the speech analysis [1]. Unfortunately, AR-model-based is considered insufficient in many cases. It is not enough for analyzing signals which contain poles such as nasal sound. For those kinds of signals the autoregressive moving average (ARMA)-based-model is considered to be more appropriate.

It has been commonly known that the nature behavior of the signals affect the accuracy of the estimation result[2]. Conventionally, the optimal models are solved by the least square method by minimizing the sum of the square of the residual signal. By doing so, all parts of the signal are assigned equal weighting function so that small parts of the signals with large amplitude has more effect to the obtained spectral estimate than that of the large parts with small amplitude. The performance of the estimator deteriorates. The obtained estimate is very much affected by large amplitude residual parts.

In the past, many efforts have been done to improve the accuracy of the AR-model-based spectral estimation. It is done by suppressing the effect of large amplitude residual signal parts. A nonlinear weighting function is applied to reduce the effect. One of the most popular nonlinear functions is derived by assuming that the residual signal is identically and independently distributed with Huber's distribution [3]. By doing so, the small portions of large amplitude errors are assigned a small weighting factor. On the other hand, large weighting factor is applied for large portions of small amplitude error signals. Recently, we proposed the usage of t-distribution assumption to reduce the effect of large amplitude for speech analysis[2]. We have shown and proved that the effect of large amplitude error is less by applying t-distribution assumption than that by utilizing the Huber's distribution assumption [2].

Extending the already proposed method in [2], in this paper we propose a robust spectral estimation method based on ARMA model to be able to analyze signals with poles and zeroes. The robustness is achieved by using the t-distribution assumption to reduce the effect of large amplitude error signals.

The proposed algorithm has been applied for analyzing synthetic signals which were generated by exciting a certain ARMA system with Gaussian and impulsive input. The simulation results show that when the Gaussian excitation is used, the accuracy of the obtained estimate by using the proposed and the conventional least square methods are comparable. On the other hand, when the excitation is impulsive, the accuracy of the estimation results can be improved by applying the proposed t-distribution assumption with small α.

2. THE PROPOSED METHOD

We consider a zero mean signal which is the output of the p-th and q-th order time invariant autoregressive moving average ARMA(p,q) system as given in equation (1). The signal is considered within a window $0 <= n <= (N-1)$. The length of the window N is assumed to be very long, $N \to \infty$. The output or the observed signal is $u(n)$. The excitation signal is $y(n)$. The parameter of the system is denoted as $a(i)$ and $b(j)$. The order of the ARMA system, p and q have to be predetermined based on the applications.

$$u(n) = y(n) + \sum_{j=1}^{q} b'(j)y(n-j) - \sum_{i=1}^{p} a'(i)u(n-i) \quad (1)$$

The transfer function of the system between the input and output is shown in equation (2).

$$H(z) = \frac{U(z)}{Y(z)} = \frac{B'(z)}{A'(z)} \quad (2)$$

where

$$A'(z) = 1 + \sum_{i=1}^{p} a'(i)z^{-i} \quad (3)$$

and

$$B'(z) = 1 + \sum_{j=1}^{q} b'(j)z^{-j} \quad (4)$$

The spectrum of the signal can be obtained be calculating $H(e^{j\omega})$ by setting $z = e^{j\omega}$. Therefore, it is necessary to estimate $a'(i)$, by

$$\mathbf{a}^T(n) = [a(1) \;\; a(2) \;\; \ldots \;\; a(p)] \quad (5)$$

and $b'(j)$ by

$$\mathbf{b}^T(n) = [b(1) \;\; b(2) \;\; \ldots \;\; b(q)] \quad (6)$$

It is done by estimating an inverse system

$$G(z) = \frac{1}{H(z)} = \frac{O(z)}{U(z)} = \frac{1+\sum_{i=1}^{p} a(i)z^{-i}}{1+\sum_{j=1}^{q} b(j)z^{-j}} \quad (7)$$

The input of the ARMA estimator is the observed signal $u(n)$ and the output is the residual signal $o(n)$.

In the conventional spectral estimation, it is assumed that the error or the residual signal is IID with Gaussian probability density function. In this case, the optimal parameter is calculated to minimize the sum of the error signal. All portion of the error signal are assigned the same weighting factor, so that large amplitude error signals have more effect on the obtained adaptation step than that of small amplitude error signals. The performance of the estimator deteriorate when the actual excitation is impulsive non Gaussian signal.

In this paper, we propose that the usage of the t-distribution with small α assumption to derive the new objective function. By doing so, we assume that the probability density function of the error signal is [4]

$$f_\alpha(x) = K_\alpha \{w_\alpha(x)\}^{1/\alpha} \quad (8)$$

where K_α is a constant depend on the degree of freedom α and it is defined as

$$K\alpha = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \quad (9)$$

and the weighting factor is

$$w_\alpha(y) = \frac{1}{1+\frac{y^2}{\alpha}} \quad (10)$$

The probability density function (PDF) $f_\alpha(x)$ is depicted in Figure 1.

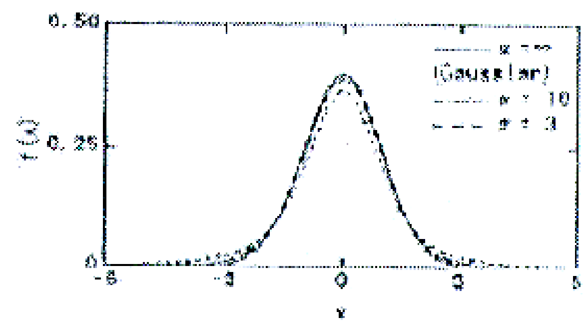

Figure 1. The probability density function (PDF) $f_\alpha(x)$ for various α.

Please note that $f_\infty(x)$ is the Gaussian distribution with unity standard deviation and zero mean $N(0,1)$ which is used in the conventional least square method. For estimation purposes, $f_\alpha(x)$ has to have a finite second moment [4]. Since $f_\alpha(x)$ for $\alpha < 3$ has an infinite second moment, in this paper we use $\alpha >= 3$. By using small α we assume that the residual is more impulsive than that in case of Gaussian assumption.

The optimal adaptive system coefficient is selected to maximize the log likelihood of the error signal in (8).

$$L = \log K\alpha - L(o(n)) \quad (11)$$

where

$$\bar{L}(o(n)) = \frac{\alpha+1}{2} E\left\{\log\left(w_\alpha\left(\frac{o(n)}{s}\right)\right)\right\} \quad (12)$$

and $E\{.\}$ is the mean operator. The robust scale estimate is calculated by

$$\hat{s} = \text{median}\,|o(n)| \quad (13)$$

along a certain specified window $0 <= n <= (scl - 1)$.

Larger scl will produce more accurate result. On the other hand, larger scl needs more calculations to solve the median. Compromizing both factors, in this paper we select a fixed $scl = 64$.

Since the degree of freedom α is a pre-selected value, the first term in (11) is a constant. Thus, maximizing the log likehood function in (11) is solved by minimizing (12) using the Newton Raphson nonlinear optimization[5]. The optimal coefficient is solved iteratively by

$$\begin{bmatrix} \mathbf{a}(n) \\ \mathbf{b}(n) \end{bmatrix}^{q} = \begin{bmatrix} \mathbf{a}(n) \\ \mathbf{b}(n) \end{bmatrix}^{q-1} - \mathbf{H}^{-1}\nabla \quad (11)$$

where the iteration number is defined as q. The gradient vector is defined as

$$\nabla(n) = \sum_{n=0}^{N-1} w(n) o(n) \begin{bmatrix} \mathbf{x}(n) \\ \mathbf{o}(n) \end{bmatrix} \quad (12)$$

and the Hessian matrix is

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 L}{\partial \mathbf{a} \partial \mathbf{a}} & \frac{\partial^2 L}{\partial \mathbf{a} \partial \mathbf{b}} \\ \frac{\partial^2 L}{\partial \mathbf{b} \partial \mathbf{a}} & \frac{\partial^2 L}{\partial \mathbf{b} \partial \mathbf{b}} \end{bmatrix} \quad (13)$$

$$= \sum_{n=0}^{N-1} \mathrm{w(n)\,x(n)}$$

$$\begin{bmatrix} \mathbf{u}(n)\mathbf{u}^T(n) & \mathbf{u}(n)\mathbf{o}^T(n) \\ \mathbf{u}(n)\mathbf{o}^T(n) & \mathbf{o}(n)\mathbf{o}^T(n) \end{bmatrix} \quad (14)$$

where the input vector is defined as

$$\mathbf{u}(n) = \mathrm{col}[u(n+i)] \quad (15)$$

and $1 <= i <= L$. We also defined the output vector as

$$\mathbf{o}(n) = \mathrm{col}[o(n+j)] \quad (16)$$

and $1 <= j <= q$. A certain initial value for $a(i)$ and $b(j)$ has to be selected to start the iteration process.

The iteration is terminated when either

$$\left| L^{q} - L^{q-1} \right| \le 0.01 \quad (17)$$

or

$$\sqrt{\sum_{i=1}^{p}\left(\frac{\partial L}{\partial a(i)}\right)^2 + \sum_{j=1}^{q}\left(\frac{\partial L}{\partial b(j)}\right)^2} \le 0.01 \quad (18)$$

is satisfied.

When we set the degree of freedom $\alpha = \infty$, we get the conventional least square approach for ARMA estimator. Therefore, the proposed ARMA estimator can be regarded as a generalization of the conventional least square method.

The simulation results in section 3 show that when the excitaion signal is impulsive non-Gaussian, by applying small α assumption we can obtain more efficient and more accurate estimate than that obtained by using large α; i.e. $\alpha=\infty$ as it is applied for the conventional least square method. Since in the real application, we have no a-priori knowledge about the nature of the excitation signal and it maybe impulse, we recommend the usage of the small α assumption to get better estimation.

3. THE SIMULATION RESULTS

The proposed estimation algorithm has been realized and applied to analyze a synthetic signal which has poles and zeroes. The synthetic signal was generated by feeding a certain excitation signal into a ARMA(5,5) system. We are intended to estimate the spectrum of the signal. We used the zero mean and unity variance random $N(1,0)$ Gaussian noise as the first excitation signal. The noise was generated by using the Box-Muller technique[5]. The spectrum of the signal was determined only from the output of system. No further information was provided into the estimator. The sampling frequency is assumed to be 10 Khz. The starting value is selected to be $\mathbf{a}^0(n) = 0$ and $\mathbf{b}^0(n) = 0$. We applied the Newton Raphson iteration method to calculate the optimal coefficient. We applied $\alpha=3$ and $\alpha=\infty$. The obtained spectral are depicted in Figure 1. Those plots show that the obtained estimate spectral by using $\alpha=3$ and $\alpha=\infty$ are almost similar with the obtained periodogram. This results show that the proposed algorithm by using small α is also appropriate when the signal is Gaussian and not containing too many impulsive parts. Therefore, the conventional Gaussian assumption is sufficient to obtained accurate estimate. The proposed algorithm does not introduce additional error. The spectral shown in Fig. 1 is the average spectral calculated from 100 frames. Each frame is 256 msec and containing 256 samples. We used the rectangular window for each frame.

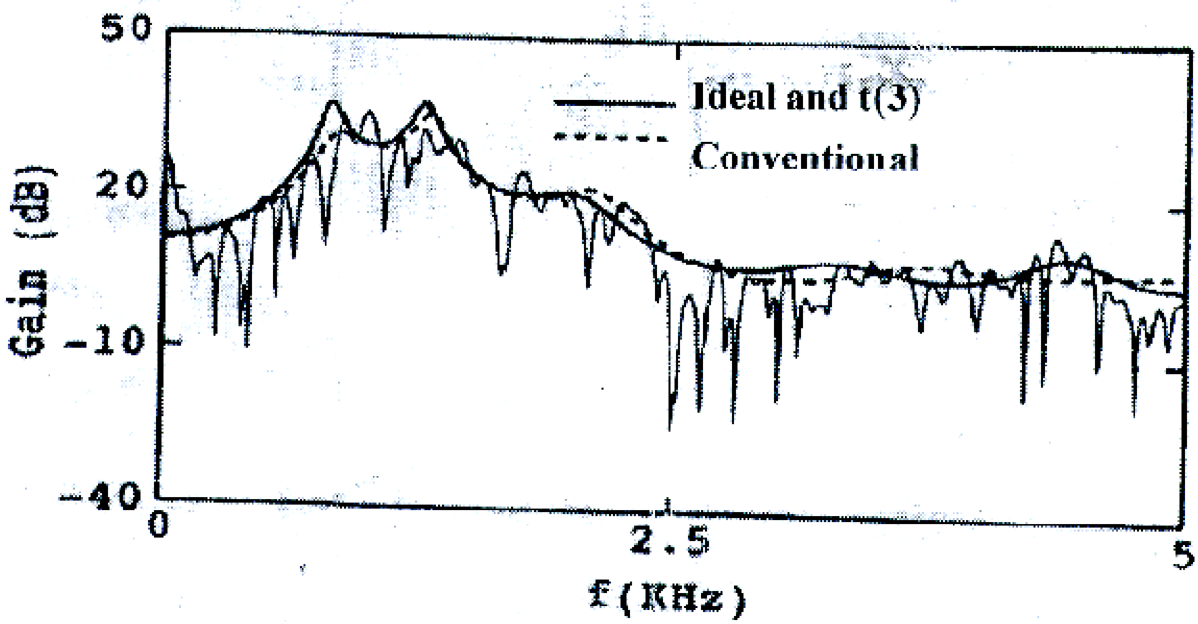

Figure 1 The ideal and the obtained estimate spectra using various estimators when the excitation is Gaussian noise.

The second synthetic signal is generated by feeding an impulsive signal into an ARMA system. The impulsive signal is generated by applying a nonlinear function

$$h(n) = \begin{cases} 1 \text{ if } g(i) \geq \delta \\ -1 \text{ if } g(i) \leq \delta \\ 0 \quad \text{elsewhere} \end{cases} \qquad (19)$$

Where $g(I)$ is the Gaussian noise. In this paper δ is arbitrary selected to be 0.75.

The ideal and the average of the obtained estimate spectra are given in Fig. 3. The average was calculated from 100 frames. The plots on Fig. 2 show that when $\alpha=\infty$ is used, there are some error in the estimated spectral.

There are some deviations from the ideal spectrum. This is because the estimator is very much affected by the large amplitude parts. This condition can be improved by applying the proposed method with $\alpha=3$.

By doing so the effect of large amplitude residual parts is reduced so that it can be seen in Fig. 2 that the error between the ideal and the estimate spectra is smaller.

Therefore, since the nature of the excitation signal is unknown a-priori, we recommend the usage of t-distribution assumption with small α for ARMA spectral estimation which has been proved to be appropriate for analyzing signal which has many large amplitude parts such as in case of impulsive excitation but also appropriate for analyzing signal which are not containing too many large amplitude parts such as in Gaussian excitation case.

We have also calculate the average error(AV) and the standard deviation (SD) of the estimated results. The AV and SD are depicted in Fig. 3. Those plots also show that the error between the ideal and the estimated spectral is smaller by applying small α than that when large α is used. The SD of the error is also smaller when small α is used. This result show that the estimated spectral is not very much affected by large amplitude residual.

4. CONCLUSIONS

We have presented a novel ARMA spectral estimator system based on t-distribution with α degrees of freedom assumption. The optimal coefficient is solved by the Newton Raphson algorithm. In case when the input is impulsive contaminated image, the error of the obtained estimate spectral is smaller than that by using the conventional least square approach. Further applications of the proposed algorithm is still studied and will be reported somewhere else.

5. REFERENCES

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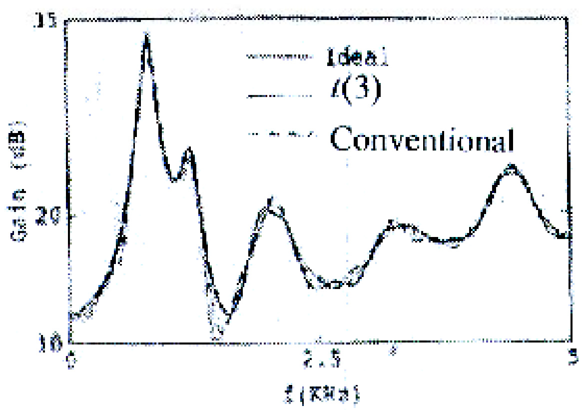

Figure 3. The ideal and estimated spectra when the excitation is impulsive.

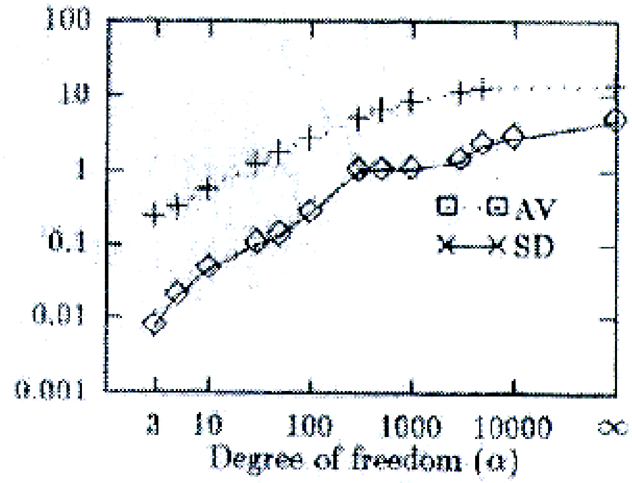

Figure 4. The average (AV) and the standard deviation of the error between the ideal and the obtained estimate spectra.