# AN APPROXIMATION TO THE VOLTERRA SERIES WITH MULTIPLE LINEAR ARMA FILTERS

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## ABSTRACT

*Nonlinear filtering based on the Volterra series expansion is a powerful universal tool in signal processing. Due to the problem of increased complexity for higher orders and filter lengths, approximations up to third order nonlinearities using linear FIR-filters and multipliers have been developed earlier, called Multi Memory Decomposition (MMD). In our paper we go a step further in this approach using ARMA-filters instead, which leads to reduction in the number of coefficients to about 50% for similar system functions. The good performance of this new approach is demonstrated by means of a processor designed for identification of nonlinear loudspeaker distortions.*

## 1. INTRODUCTION

Structures, based on the Volterra series expansion [5], are widely known and used for modeling and compensation of nonlinear systems. The Volterra series expansion uses a nonlinear combination of past input samples *x*, which are multiplied and weighted by the respective coefficient matrices. A general form of the filter output *y* can be written as

$$y[n] = \sum_{\nu_1=0}^{N-1} h_1[\nu_1]x[n-\nu_1] + \ldots + \sum_{\nu_1,\ldots,\nu_p=0}^{N-1}\cdots\sum h_p[\nu_1,\ldots,\nu_p]x[n-\nu_1]\cdots x[n-\nu_p]\,, \qquad (1)$$

where $h_p$ is the *p*-dimensional coefficient matrix and *N* the memory length.

The number of coefficients in (1) is of the order $N^p$, or short, $O(N^p)$. To reduce the number used by this general structure, approximations have been developed. One of these is the Multi Memory Decomposition (MMD) [1], [2], which has been shown to have excellent approximation properties. This structure, which is shown in Figure 1 up to cubic order, consists only of linear FIR-filters and multipliers. In this case it produces nonlinearities up to 3rd order whereas the complexity in calculation is comparable to linear filtering, $O(N)$.

The filter output is obtained as

$$y[n] = \sum_{i=0}^{N_1-1} a[i]x[n-i] \quad +$$
$$\sum_{i=0}^{N_2^p-1} b_p[i] \sum_{j=0}^{N_2^a-1} b_1[j]x[n-i-j] \sum_{k=0}^{N_2^a-1} b_1[k]x[n-i-k] +$$
$$\sum_{i=0}^{N_3^p-1} c_p[i] \sum_{j=0}^{N_3^a+N_3^m-2} c_3[j]x[n-i-j] \sum_{k=0}^{N_3^m-1} c_m[k]\cdot$$
$$\sum_{l=0}^{N_3^a-1} c_1[l]x[n-i-l-k] \sum_{m=0}^{N_3^a-1} c_2[m]x[n-i-m-k], \qquad (2)$$

where *a* is the coefficient vector of the linear part and $b_i$ and $c_i$ are the weight vectors of the quadratic and cubic part of the model, respectively.

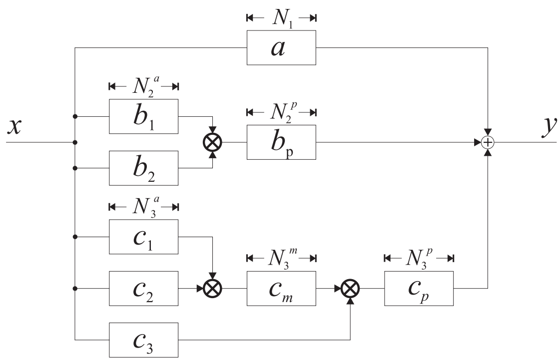


**Figure 1.** Approximation of a 3rd order Volterra series expansion with a MMD-filter structure

Given this structure, the properties of the nonlinear system are determined solely by the transfer functions of the linear filters used. By replacing each FIR-filter by an appropriate ARMA-filter with reduced length, hence, we can recreate the system with a substantial reduction of coefficients. In the following, we will call this modification MMD-ARMA to distinguish from the standard MMD-model.

## 2. THE MMD-ARMA STRUCTURE

As mentioned above, each FIR-filter of the MMD-model according to Figure 1 is now replaced by an IIR- or ARMA-filter (See also Figure 2).

The exact description of the filter output for the first three orders is given by

$$y[n] = y_l[n] + y_q[n] + y_c[n]\,, \tag{3}$$

where $y_l[n]$, $y_q[n]$ and $y_c[n]$ are defined by

$$y_l[n] = \sum_{i=0}^{N_1-1} a_{ma}[i]x[n-i] - \sum_{i=1}^{N_1-1} a_{ar}[i]y_l[n-i] \tag{4}$$

for the linear part,

$$y_q[n] = \sum_{i=0}^{N_2^p-1} b_{pma}[i] \cdot \left(\sum_{j=0}^{N_2^a-1} b_{1ma}[j]x[n-i-j] - \sum_{j=1}^{N_2^a-1} b_{1ar}[j]y_{q1}[n-i-j]\right) \left(\sum_{k=0}^{N_2^a-1} b_{2ma}[k]x[n-i-k] - \sum_{k=1}^{N_2^a-1} b_{2ar}[k]y_{q2}[n-i-k]\right) \sum_{i=1}^{N_2^p-1} b_{par}[i]y_q[n-i] \tag{5}$$

for the quadratic part and (6) at the end of this page for the cubic part. The respective indices for *y* in these equations denote the specific output of each linear filter, specified by the index of the respective coefficient vectors, which can also be found in Figure 2.

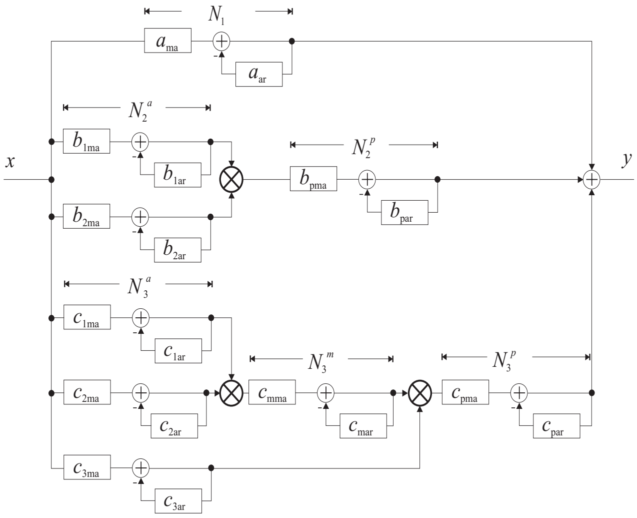


**Figure 2.** MMD-ARMA filter structure based on linear ARMA filters and multipliers

Hence the recursive portions of the structure are situated only in the linear filter parts the global structure itself remains nonrecursive which is significant for several reasons. First, the linear part and the single orders of nonlinearities are independent of each other - allowing independent optimization e.g. -, which is different to recursive Volterra structures as presented e.g. in [4]. Next, assuming that each linear filter is stable which is straightforward from the deduction of the FIR-structure, the whole system will remain stable. The new structure takes the advantage of recursive

$$y_c[n] = \sum_{i=0}^{N_3^p-1} c_{pma}[i] \left[ \sum_{j=0}^{N_3^m+N_3^a-2} c_{3ma}[j]x[n-i-j] - \sum_{j=1}^{N_3^m+N_3^a-2} c_{3ar}[j]y_{c3}[n-i-j] \right] \left[ \sum_{k=0}^{N_3^m-1} c_{mma}[k] \left( \sum_{l=0}^{N_3^a-1} c_{1ma}[l]x[n-i-l-k] - \sum_{l=1}^{N_3^a-1} c_{1ar}[l]y_{c1}[n-i-l-k] \right) \left( \sum_{m=0}^{N_3^a-1} c_{2ma}[m]x[n-i-m-k] - \sum_{m=1}^{N_3^a-1} c_{2ar}[m]y_{c2}[n-i-m-k] \right) - \sum_{k=1}^{N_3^m-1} c_{mar}[k]y_{cm}[n-i-k] \right] - \sum_{i=1}^{N_3^p-1} c_{par}[i]y_c[n-i] \tag{6}$$

filters, in particular by reducing the number of coefficients for the approximated Volterra series expansion compared to the FIR-filter solution.

## 3. CALCULATION OF COEFFICIENTS BASED ON EXISTING MMD-COEFFICIENTS

If the desired system is already modeled as an MMD-system, e.g. with one of the methods described in [1] and [2], the corresponding MMD-ARMA-coefficients can be computed easily by using standard transformations between MA and ARMA coefficients. So each ARMA filter is designed from the impulse or frequency response of the respective MA filter. Methods for this purpose are wellknown and straightforward and will not be discussed further. However, this approach can only be used in offline algorithms.

## 4. CALCULATION OF COEFFICIENTS BASED ON INPUT/OUTPUT SIGNAL

Given the input and output signal of the desired system, the MMD-ARMA coefficients can be obtained directly by minimizing the residual error $e[n]$, defined as the difference between the desired output $d[n]$ and the MMD-ARMA filter output $y[n]$, as shown in Figure 3.
Similarly to [1] and [2] the output of the system, respectively equations (4), (5) and (6) can also be written in vector notation form, depending on coefficient and input vectors. So the determination of coefficients can be done by implementing an adaptive LMS algorithm which implies the linearization of the structure. This means keeping 2 resp. 4 vectors for the 2$^{nd}$ resp. 3$^{rd}$ order system constant while adapting the others. In order to do this, state vectors and matrices have to be introduced which comprise the information of signal and remaining coefficients. The resulting equations now written in form of a linear convolution can be solved by standard LMS algorithms.
Despite of the linearity of the algorithms the error surface of this system is not quadratic. Therefore, the adaption can get stuck in a local minimum. This, however, turns out not to be a serious problem in most applications as simulations have shown that - at least for the various signals we used - the performance of different minima is nearly the same and the residual error differs only slightly.
Further, with this method and structure it is theoretically possible to get instable filters. In this case more sophisticated algorithms including additional stability criterions are required.

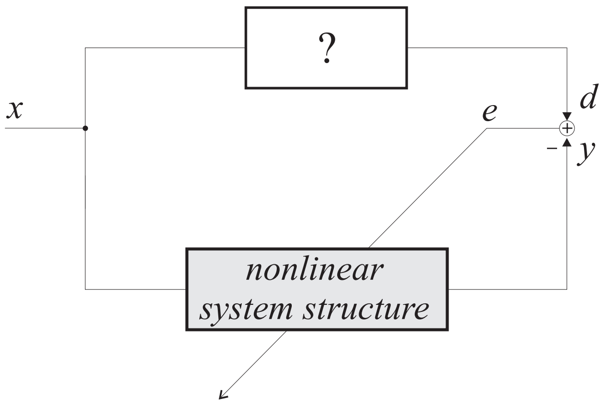


**Figure 3.** Structure for identifying nonlinear systems

## 5. APPLICATION

In order to compare the performance of this new approach to the previous one, regarding the number of coefficients needed, simulations have been performed. For this purpose both systems were required to model the nonlinear transfer function of a loudspeaker, given by a real recorded input/output data set. The result of one simulation series is shown in Figure 4, where the number of weights respective required calculations has been increased stepwise for both filter structures to find an optimum between effort of coefficients and achieved residual mean squared error. It is obvious that, looking at the MMD-ARMA approximation, for shorter filter lengths the increment of the linear part results in large steps for the residual error. Beyond a particular level of residual error, no remarkable improvement can be seen for both structures.

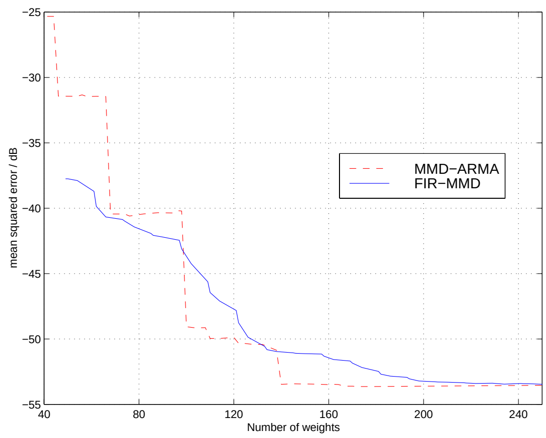


**Figure 4.** Comparison of effort between FIR-MMD and MMD-ARMA. Basically the MMD-ARMA structure needs a lesser number of filter weights.

Given a distinct value of performance, the exact filter lengths and number of coefficients of both system structures are summarized in table 1. The residual mean squared error achieved by both structures is about –53.5 dB.

**Table 1.** comparison of filter lengths and number of coefficients of FIR-MMD and MMD-ARMA. The MMD-ARMA structure needs significantly less coefficients for a given performance.

| | FIR-MMD | | MMD-ARMA | |
|---|---|---|---|---|
| | filter length | number of weights | filter length | number of weights |
| linear part | $N_1$=57 | 57 | $N_1$=10 | 20 |
| quadratic part | $N_2^a$=22, $N_2^p$=21 | 65 | $N_2^a$=8, $N_2^p$=6 | 41 |
| cubic part | $N_3^a$=19, $N_3^m$=19, $N_3^p$=18 | 112 | $N_3^a$=8, $N_3^m$=7, $N_3^p$=5 | 79 |
| total | | 234 | | 140 |

## 6. CONCLUSION

In the simulations above we have clearly demonstrated that the MMD-ARMA structure needs significantly less coefficients to achieve similar results in residual error than the well proofed FIR counterpart. The usage of recursive structures for nonlinear problems is very interesting, so future researches could examine the abilities of IIR Volterra filters, similar to [4], and approximations to it.

## 7. REFERENCE


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