DECISION FEEDBACK EQUALIZATION FROM AN H^{∞} PERSPECTIVE

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ABSTRACT

We approach the nonlinear decision feedback equalization problem from an H^{∞} estimation point of view. Using the standard (and simplifying) assumption that all previous decisions are correct, we obtain an explicit parameterization of all H^{∞} optimal decision feedback equalizers. In particular, under the above assumption, we show that MMSE decision feedback equalizers are H^{∞} optimal. The H^{∞} approach also suggests a method for dealing with errors in previous decisions, and results in an equalization scheme with improved performance over other equalizers that do not take such errors into account.

1. INTRODUCTION

One of the ultimate goals of digital communications is the ability to transmit information from one point to another with the highest possible data rates. One major obstacle in achieving this goal is intersymbol interference (ISI), which is the effect of previous and future symbols on the current symbol imposed by the communications channel, since it affects the bit error rate (BER) performance in detecting the original transmitted sequence. Therefore, various methods have been developed to increase the system performance by reducing the effects of the ISI.

Decision feedback equalization (DFE) is a nonlinear method where old decisions are employed, in conjunction with the observations, to improve the equalizer performance. The DFE has been a focus of research for more than two decades. The reference paper [1] provides a good overview and historical summary of these research efforts. A more recent treatment of minimum mean square error decision feedback equalization is in [2].

Recently, the H^{∞} criterion has been proposed [3, 4] for the linear equalization problem, with the belief that the resulting H^{∞} equalizers will be more robust with respect to model uncertainties and the lack of statistical knowledge of the exogenous signals. In this paper, we approach the decision feedback equalization problem from the H^{∞} estimation point of view. In the first part of the paper, we describe the decision feedback equalization problem. Following this section, we summarize some results from H^{∞} estimation theory that are relevant to the equalization problem. Then, in Section 4.2, we look at the H^{∞} decision feedback equalizers under the assumption that the previous decisions input to the feedback filter are correct and, among other results, show that MMSE equalizers are H^{∞}-optimal. In the last part of the paper, we abandon the assumption of the correctness of previous decisions, a situation that, in fact, is well-suited to the H^{∞} framework. We show that we can improve the BER performance with this new approach.

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2. DECISION FEEDBACK EQUALIZATION PROBLEM

The standard discrete-time model for the decision feedback equalization problem is illustrated in Figure 1. In this figure, $\{b_i\}$ represents the discrete-time finite-alphabet input data sequence. If we assume the number of users to be M then $b_i \in \mathcal{C}^M$. The distortion effects of the communications medium are represented by the linear time-invariant transfer matrix $H(z)$. If there are N antennas or branches at the receiver, *i.e.*, $y_i \in \mathcal{C}^N$, then $H(z)$ is assumed to be a causal and stable $N \times M$ matrix function. We also assume that the number of users is less than or equal to the number of antennas, *i.e.*, $M \leq N$. The sequence $\{v_i\}$ represents the noise disturbance (*e.g.*, receiver antenna noise, co-channel interference, etc.) corrupting the observations. Modeling errors due to imperfect knowledge of the true channel can also be incorporated into the additive disturbance $\{v_i\}$.

Referring to Figure 1, the basic aim in decision feedback equalization is to design $K_1(z)$, the causal feedforward filter, and $K_2(z)$, the causal feedback filter, so as to estimate b_{i-d}, where $d \geq 0$ is a parameter indicating the delay in estimating the transmitted sequence. The estimate, denoted by $\hat{b}_{i-d}$, is the sum of the outputs of $K_1(z)$ and $K_2(z)$. The design of $K_1(z)$ and $K_2(z)$ depends upon the criterion chosen to define the closeness of $\hat{b}_{i-d}$ to b_{i-d}.

Most of the research in the decision feedback area is focused on the mean square error criterion (see for ex-

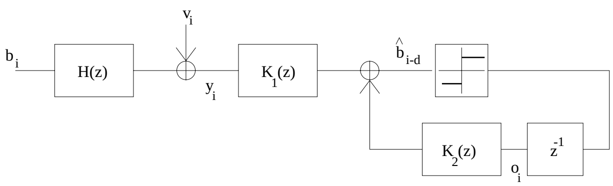

Figure 1: Decision Feedback Equalization

ample [2]), mostly because it allows for the derivation of explicit formulas for both the feed-forward and feedback filters. In this paper, we will use the H^∞ criterion as the basis of our formulation of the filters.

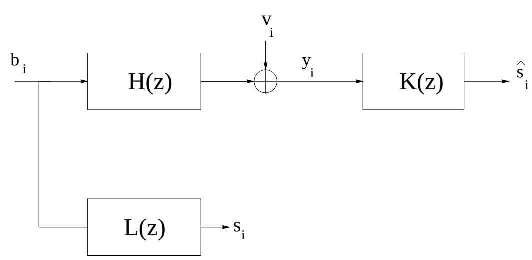

Figure 2: Setup for linear estimation

3. H^∞ ESTIMATION

The basic setup for a general linear estimation problem is illustrated in Figure 2. Here $H(z)$ and $L(z)$ are *known* causal linear time-invariant filters that map the input sequence $\{b_i\}$ to their respective outputs $\{y_i\}$ and $\{s_i\}$. The driving input $\{b_i\}$ and the additive disturbance sequence $\{v_i\}$ are assumed to be *unknown*. The estimation problem is to design a causal linear time-invariant estimator $K(z)$ that estimates the unobservable signal sequence $\{s_i\}$, using the observations $\{y_j, j \leq i\}$. We shall denote such estimates by $\hat{s}_{i|i}$ and the resulting estimation errors by $\tilde{s}_{i|i} = s_i - \hat{s}_{i|i}$. Let $T_K(z)$ denote the transfer matrix that maps the normalized unknown disturbances $Q^{-1/2}\{b_i\}$ and $R^{-1/2}\{v_i\}$ to the estimation errors $\{\tilde{s}_{i|i}\}$. Thus,

$$T_K(z) = \begin{bmatrix} (L(z) - K(z)H(z))Q^{1/2} & -K(z)R^{1/2} \end{bmatrix} \quad (1)$$

where $Q = Q^{1/2}Q^{*/2} > 0$ and $R = R^{1/2}R^{*/2}$ are positive normalization weights.

The choice of $K(z)$, and thereby the estimates $\hat{s}_{i|i}$, depends upon our choice of performance criterion. In H^∞ estimation $K(z)$ is chosen to minimize the maximum energy gain of $T_K(z)$, also known as the H^∞ norm of $T_K(z)$, defined as

$$\|T_K(z)\|_\infty^2 = \sup_{b,v \in l^2, b,v \neq 0} \frac{\sum_{i=-\infty}^{\infty} |\tilde{s}_{i|i}|^2}{\sum_{i=\infty}^{\infty} b_i^T Q^{-1} b_i + \sum_{i=-\infty}^{\infty} v_i^T R^{-1} v_i} \quad (2)$$

Problem 1 (Optimal H^∞ Filtering Problem) *:*
Find a causal estimator $K(z)$ that satisfies

$$\inf_{K(z)} \|T_K(z)\|_\infty^2 . \quad (3)$$

Moreover, find the min-max energy gain γ_{opt}^2.

There are very few cases where a closed-form solution to the optimal H^∞ filtering problem can be found, and in general one relaxes the minimization and settles for a suboptimal solution.

Problem 2 (Suboptimal H^∞ Filtering Problem) *:*
Given $\gamma > 0$, find, if possible, a causal estimator $K(z)$ that guarantees

$$\|T_K(z)\|_\infty^2 < \gamma^2 \quad (4)$$

This clearly requires checking whether $\gamma > \gamma_{opt}$.

It will now be useful to give some flavor of the solution to Problem 2. (See [5] for more details). We introduce the following so-called *Popov function*,

$$\Sigma(z) = \begin{bmatrix} R + H(z)QH^*(z^{-*}) & -H(z)QL^*(z^{-*}) \\ -L(z)QH^*(z^{-*}) & -\gamma^2 I + L(z)QL^*(z^{-*}) \end{bmatrix},$$

which can be regarded as a certain indefinite generalization of "the spectral density function". Then it can be shown that a causal estimator, $K(z)$, that achieves $\|T_K(z)\|_\infty < \gamma$ exists if, and only if, the Popov function admits a canonical J-spectral factorization of the form

$$\Sigma(z) = \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ L_{12}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{bmatrix} \quad (5)$$

with $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ and $L_{11}(z)$ causal and causally invertible, and $L_{12}(z)$ strictly causal. If this is the case, then all possible H^∞ estimators of level γ are given by

$$K(z) = (L_{22}(z)C(z) - L_{21}(z))\,(L_{11}(z) - L_{12}(z)C(z))^{-1}, \quad (6)$$

where $C(z)$ is any causal and strictly contractive operator, *i.e.*, $C(z)$ is causal and is such that

$$|C(e^{j\omega})|^2 < 1 \quad (7)$$

, for all $\omega \in [0, 2\pi]$. An important special choice is $C = 0$ which leads to the so-called "central" filter

$$K_{cen}(z) = -L_{21}(z)L_{11}^{-1}(z). \quad (8)$$

4. H^∞ DECISION FEEDBACK EQUALIZATION

4.1. An Equivalent Model

Decision feedback equalization is a nonlinear scheme due to the nonlinear decision device involved in the structure. It is therefore difficult to obtain explicit expressions for

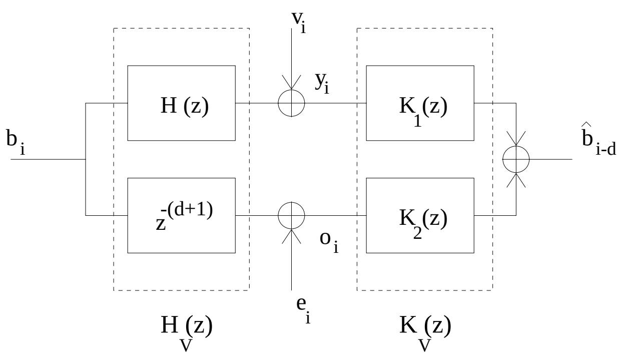

Figure 3: Equivalent Channel Model

the filters employed with respect to any criterion. By removing the nonlinear decision device, we can remodel the decision feedback equalization problem as shown in Figure 3, so that it takes the form of a general linear estimation problem with $L(z) = z^{-d}I$ (since we are trying to estimate the input symbols with a delay of d). In this figure, o_i represents the possibly incorrect decisions and e_i represents the corresponding decision errors. Thus, in this model

$$H_V(z) = \begin{bmatrix} H(z) \\ z^{-(d+1)}I \end{bmatrix} \quad (9)$$

is the equivalent matrix channel. Again according to the same model

$$Y_V(z) = \begin{bmatrix} Y(z) \\ O(z) \end{bmatrix} \text{ and } N_V(z) = \begin{bmatrix} V(z) \\ E(z) \end{bmatrix} \quad (10)$$

are the equivalent observation and noise vectors, respectively. We define the energy weighting matrix for the noise N_V as

$$R_{N_V} = \begin{bmatrix} R & 0 \\ 0 & \epsilon I \end{bmatrix}. \quad (11)$$

where R represents the weighting we assign to the additive disturbance $\{v_i\}$ and ϵ represents the weighting assigned to the decision errors $\{e_i\}$. Then the decision feedback equalization problem is equivalent to the filtering problem of finding a causal

$$K_V(z) = \begin{bmatrix} K_1(z) & K_2(z) \end{bmatrix} \quad (12)$$

that minimizes the H^∞ norm of the transfer matrix

$$T_{K_V}(z) = \begin{bmatrix} \left(z^{-d}I - K_V(z)H_V(z)\right)Q^{1/2} & -K_V(z)R_{N_V}^{1/2} \end{bmatrix}. \quad (13)$$

Therefore the Popov function for this problem is

$$\Sigma(z) = \begin{bmatrix} R_{N_V} + H_V(z)QH_V^*(z^{-*}) & -H_V(z)Qz^d \\ -z^{-d}QH_V^*(z^{-*}) & Q - \gamma^2 I \end{bmatrix}. \quad (14)$$

4.2. H^∞ Equalizers for $d = 0$

Following the common practice in the literature to simplify the derivations, we assume that the decisions input to the feedback filter $K_2(z)$ are correct (*i.e.*, $e_i = 0$). We also look at the special case where $d = 0$ and assume binary antipodal signaling for which $b_i \in \{-1, 1\}$ and therefore $Q = I$. The results can be easily generalized to more complex signal constellations. In [6], we show that

$$\gamma_{opt}^2 = (1 + \sigma_{min}(h_0^* R^{-1} h_0))^{-1} \quad (15)$$

where $h_0 = H(\infty)$ is the impulse response matrix at zero delay. **Remarks:**

- It is interesting to compare the performance of the H^∞ decision feedback equalizer with the H^∞ linear equalizer by comparing the corresponding optimal H^∞ norms. As shown in [3], for a scalar channel, the linear equalizer has

$$\gamma_{opt,linear}^2 = \frac{r}{r + \min_w |H(e^{jw})|^2} \quad (16)$$

for a *minimum phase* $H(z)$ and

$$\gamma_{opt,linear}^2 = 1 \quad (17)$$

for a *non-minimum phase* $H(z)$. For the decision feedback equalizer, irrespective of the minimum phase property of the channel, (15) yields

$$\gamma_{opt,dfe}^2 = \frac{r}{r + |h_0|^2} \quad (18)$$

For non-minimum phase channels, it is clear that $\gamma_{opt,dfe}^2 \leq \gamma_{opt,linear}^2$ since $|h_0|^2 \geq 0$ and therefore $\gamma_{opt,dfe}^2 \leq 1$. For minimum phase channels, over the region $|z| \geq 1$, the minimum value of the $H(z)$ is achieved on the unit circle. This is due to the observation that $H^{-1}(z)$ has its all poles inside the unit circle and therefore, by the maximum modulus theorem, $H^{-1}(z)$ achieves its maximum on the unit circle for $|z| \geq 1$.Thus $H(z)$ achieves its minimum on the unit circle for this region . Since $h_0 = H(\infty)$, we have

$$|h_0|^2 \geq min_w |H(e^{jw})|^2, \quad (19)$$

so that,

$$\gamma_{opt,dfe}^2 \leq \gamma_{opt,linear}^2 \quad (20)$$

i.e. the the H^∞ decision feedback equalizer has better H^∞ performance than the H^∞ linear equalizer.

- Another important observation is obtained when we look at the central solution (8) to the decision feedback equalization problem (see [6] for details):

$$
\begin{aligned}
K_{central}(z) &= -L_{21}(z)L_{11}^{-1}(z) && (21)\\
&= \begin{bmatrix} h_0^*(h_0h_0^*+R)^{-1} \\ -(h_1+h_2z^{-1}+\ldots)h_0^*(h_0h_0^*+R)^{-1} \end{bmatrix}^T
\end{aligned}
$$

  which turns out to be the MMSE decision feedback equalizer (MMSE-DFE) for the given setup (and further assuming that $\{b_i\}$ is an independent Bernoulli process with parameter $\frac{1}{2}$ and that $\{v_i\}$ is white noise with covariance R).

- The factorization of the Popov function (14) can be achieved easily, but with increasing complexity of the expressions, for any $d > 0$. This is because the Popov function, under the correct decisions assumption, *i.e.*, $\epsilon = 0$, is always unimodular [6].

5. H^∞ OPTIMALITY OF MMSE DECISION FEEDBACK EQUALIZATION

An important result from the previous section is that under the correct previous decisions assumption, and for $d = 0$, the MMSE decision feedback equalizer is H^∞-optimal. It turns out that this is true for any $d > 0$. To show this fact, we can look at the error spectrum corresponding to the MMSE-DFE which is given by

$$
\begin{aligned}
Er(z) = \quad & \{z^{-d}H_V^*(z^{-*})M^*(z^{-*})\}_- \\
& \{\{z^{-d}H_V^*(z^{-*})M^*(z^{-*})\}_-\}^* \qquad (22)
\end{aligned}
$$

where $\{\cdot\}_-$ denotes the strictly non-causal part of its argument and $\{A(z)\}^* \stackrel{\Delta}{=} A^*(z^{-*})$ for any function $A(z)$. $M(z)$ is the causal and causaly invertible factor obtained from the canonical factorization of $S_{Y_V}(z)$ which can be written as

$$
\begin{aligned}
S_{Y_V}(z) &= R_{N_V} + H_V(z)H_V^*(z^{-*}) && (23)\\
&= \begin{bmatrix} R+H(z)H^*(z^{-*}) & H(z)z^{d+1} \\ z^{-d-1}H^*(z^{-*}) & I \end{bmatrix} && (24)\\
= & \underbrace{\begin{bmatrix} R^{1/2} & H(z)z^{d+1} \\ 0 & I \end{bmatrix}}_{N(z)} \underbrace{\begin{bmatrix} R^{1/2} & 0 \\ H^*(z^{-*})z^{-d-1} & I \end{bmatrix}}_{N^*(z^{-*})} && (25)\\
& && (26)
\end{aligned}
$$

The canonical factor $M(z)$ can be writen in terms of $N(z)$ as

$$
\begin{aligned}
M(z) &= N(z)\Theta(z) && (27)\\
= & \begin{bmatrix} R^{1/2} & H(z)z^{d+1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{bmatrix} && (28)\\
& && (29)
\end{aligned}
$$

where $\Theta(z)\Theta^*(z^{-*}) = I$ is chosen such that $M(z)$ is causal and causally invertible. The causality of the $M(z)$ constrains $\theta_{21}(z)$ and $\theta_{22}(z)$ to be causal.

In [7], it is shown that the error spectrum is frequency independent and equal to

$$
Er(z) \quad = \quad |\theta_{21}(\infty)|^2 + |\theta_{22}(\infty)|^2. \qquad (30)
$$

In the scalar case, a flat error spectrum implies the simultaneous minimization of both the maximum value of the error spectrum and its area under the curve. Since the MMSE-DFE minimizes the area under the curve, this implies that it is also H^∞-optimal. We show in [7] that this property of the MMSE-DFE can also be extended to more general matrix channels. This is a striking result, which sheds further light on the properties of the MMSE-DFE. Moreover, it is a rare case, if not the only non-trivial one, where the solutions to the H^∞ and MMSE filtering problems coincide.

6. ERROR IN THE FEEDBACK

In the previous sections, to simplify the analysis and the derivation of the filters, we made the standard assumption that the decisions used by the feedback filter were always correct. When the channel is known a priori, this assumption can be made to hold by using precoding techniques (such as the Tomlinson-Harashima precoder [8]) that implement the feedback part in the transmitter section. However, when the channel is not known a priori (as is often the case) the existence of decision errors is inevitable.

When we have decision errors the $\{e_i\}$ form some non-zero sequence. Since $\{e_i\}$ is a complicated function of the feed-forward and feedback filters, as well as the other parameters in the system, it is almost impossible to give an explicit statistical description of the errors and to therefore design a filter with respect to some statistical criterion such as MMSE.

However, as far as the H^∞ criterion is concerned, $\{e_i\}$ is an unknown non-zero sequence of small power given by, say, ϵ. Therefore the H^∞ criterion can provide a solution which safeguards against the worst-case decision errors. Thus, we propose to design an H^∞-optimal equalizer that minimizes the maximum energy gain from the disturbances $Q^{-1/2}\{b_i\}$, $R^{-1/2}\{v_i\}$, and $\{\epsilon^{-1/2}e_i\}$ to the estimation errors $\{\tilde{s}_{i|i}\}$.

When $\epsilon \neq 0$, the corresponding Popov function (14) is no longer unimodular and therefore we cannot obtain explicit expressions for the H^∞ equalizers for this case. However, we can still obtain numerical solutions (to desired accuracy) by solving Riccati equations (or recursions). An important conclusion is that, when $\epsilon \neq 0$, the H^∞-optimal solution is *different* from the corresponding MMSE-DFE, which assumes a white decision error sequence [6].

In the design of the filters, we need to choose the γ and ϵ parameters. The γ should be clearly be greater than γ_{opt}. Although there is no explicit expression for γ_{opt}, one can use the upper bound provided in [7].

The choice of the ϵ parameter is critical since it represents the power of the decision errors, which is not known beforehand. Figure 4 illustrates the variation of the BER of the central H^∞ equalizer as a function of ϵ for the channel $H(z) = 0.56 - 0.06z^{-1} + 1.07z^{-2} + 1.6z^{-3} - 0.13z^{-4}$, with delay $d = 2$ and $SNR = 18dB$. As we increase the value of ϵ from 0, the BER decreases (since the equalizer is beginning to take the decision errors into account) until we reach a minimum point. After this point the BER begins to increase because ϵ overestimates the actual decision error power.

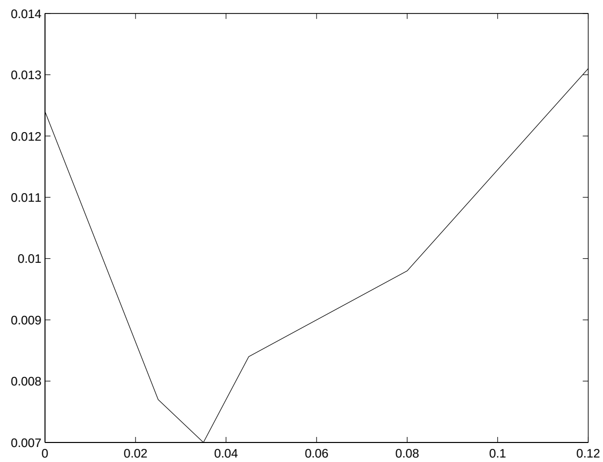

Figure 4: BER vs. ϵ

In Figure 5, we have compared the performances of the H^∞-DFE for the $\epsilon = 0$ case and the choice of ϵ that minimizes the BER, for the same channel and delay as above. The dashed line refers to the BER vs. SNR performance of the MMSE-DFE that assumes correct decisions and therefore uses $\epsilon = 0$, whereas the solid line represents the performance of the H^∞-DFE which has ϵ chosen to minimize the BER. As can be observed from the figure, depending on the SNR, the use of the ϵ parameter can provide 1dB to 2dB gain.

7. CONCLUSION

We studied the problem of decision feedback equalization from an H^∞ estimation point of view. We can convert nonlinear decision feedback scheme to a linear one by replacing nonlinear decision path with delayed input corrupted by decision error sequence. It is hard to give a useful model for the decision error sequence, therefore, it is generally assumed to be zero. We provided with the H^∞ formulation of the decision feedback equalizers under that assumption and showed that MMSE and H^∞ solutions coincide. For the more realistic case where decision errors are not assumed to be zero, it is hard to formulate DFE filters with respect to a statistical criterion, such as

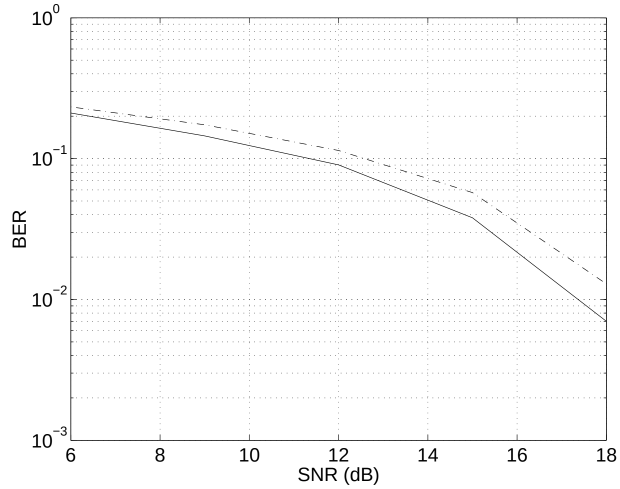

Figure 5: BER vs. SNR

MMSE, but the H^∞ framework still provides a solution with better performance than the MMSE-DFE filter that does not take these errors into account. As a result, H^∞ approach can be considered to be worst-case compensation for the errors introduced by linear approximation of the nonlinear structure.

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