NONLINEAR FILTERS WITH SYMMETRY CONSTRAINTS

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ABSTRACT

Nonrecursive digital filters are popular because they are inherently stable, unlike their recursive counterparts, and have been shown to be effective in a wide variety of situations. For example, Astola and Kuosmanen describe 20 different classes of nonrecursive filters [2], many of which are included in the general class of nonlinear smoothing filters considered by Mallows [11]. This paper defines three classes of nonrecursive filters based on symmetry restrictions and explores some of the consequences of these restrictions. Specific examples of each class are presented, general procedures are given for constructing new filters in each class from known examples, and the influence of these symmetries on root sequences is examined.

1. PROBLEM FORMULATION

Consider the moving average filter of width $2K+1$ centered at k defined by

$$y_k = \Phi(x_{k-K}, \ldots, x_k, \ldots, x_{k+K}) \equiv \Phi(\mathbf{w}_k) \quad (1)$$

and define $\mathbf{J}$ as the $(2K+1) \times (2K+1)$ permutation matrix with 1's on the cross-diagonal (lower left to upper right). This paper considers the following three filter classes:

$\mathcal{P}_K$: $\Phi(\mathcal{P}\mathbf{w}_k) = \Phi(\mathbf{w}_k)$ for all permutations $\mathcal{P}$

$\mathcal{R}_K$: $\Phi(\mathbf{J}\mathbf{w}_k) = \Phi(\mathbf{w}_k)$

$\mathcal{C}_K$: $\Phi(\mathbf{J}\mathbf{w}_k) = -\Phi(\mathbf{w}_k)$.

Note that $\mathbf{J}\mathbf{w}_k$ represents a time-reversal of the data in $\mathbf{w}_k$, motivating the notation $\mathcal{R}_K$, and matrices $\mathbf{A}$ satisfying the condition $\mathbf{JAJ} = -\mathbf{A}$ are called centroskew, motivating the notation $\mathcal{C}_K$. Since filters in class $\mathcal{P}_K$ are invariant under all permutations of the data window $\mathbf{w}_k$ and filters in class $\mathcal{R}_K$ are invariant under the specific permutation $\mathbf{J}$, it follows immediately that $\mathcal{P}_K \subset \mathcal{R}_K$. Conversely, note that if $\Phi \in \mathcal{R}_K \cap \mathcal{C}_K$, then $\Phi(\mathbf{w}_k) = -\Phi(\mathbf{w}_k) = 0$ for all $\mathbf{w}_k$. Overall, the classes $\mathcal{P}_K$, $\mathcal{R}_K$ and $\mathcal{C}_K$ include many popular filter classes, either entirely or in part.

2. THE FILTER CLASS $\mathcal{P}_K$

The class $\mathcal{P}_K$ consists of the FIR filters of the form (1) that are symmetric with respect to all of the arguments x_{k-j} in the data window $\mathbf{w}_k$. Note that this class includes all L-filters with constant weights $\{w_i\}$:

$$y_k = \sum_{i=-K}^{K} w_i x_{(i)}, \quad (2)$$

where $x_{(i)}$ are the rank-ordered data values:

$$x_{(-K)} \leq \cdots \leq x_{(0)} \leq \cdots \leq x_{(K)}. \quad (3)$$

In this notation, the median filter corresponds to $y_k = x_{(0)}$ for any window width K.

This symmetry restriction is closely related to the statistical notion of *exchangeability* [4, 5, 10]: a finite sequence of random variables is said to be exchangeable if the joint distribution is invariant under all possible permutations. This notion may be extended to infinite sequences $\{x_k\}$ if all finite subsequences are exchangable, and the cannonical example of an exchangable sequence is an independent, identically distributed (i.i.d.) sequence of random variables. Not surprisingly, since this working assumption is widely invoked in classical statistics, most of the "standard" statistical characterizations of data sequences are invariant under all data permutations. Hence, symmetric moving window implementations of these estimators lead immediately to moving average filters of class $\mathcal{P}_K$.

It is also interesting to note that this class of moving average filters is essentially nonlinear since the only linear members of $\mathcal{P}_K$ are of the form $y_k = A\bar{\mathbf{w}}_k$ for some constant A, where $\bar{\mathbf{w}}_k$ is the unweighted arithmetic mean of the data window. As a specific nonlinear example, consider the second-order Volterra filter

$$y_k = y_0 + \mathbf{a}^T\mathbf{w}_k + \mathbf{w}_k^T\mathbf{B}\mathbf{w}_k. \quad (4)$$

Here, $\mathbf{a} = A\mathbf{e}$ where $\mathbf{e} = [1, \ldots, 1]^T$ and $\mathbf{B}$ is a symmetric matrix that must satisfy $\mathbf{P}^T\mathbf{B}\mathbf{P} = \mathbf{B}$ for all permutations $\mathbf{P}$. It has been shown [4, Thm. 8] that this condition is satisfied if and only if

$$\mathbf{B} = \alpha\mathbf{Q} + \beta\mathbf{E} \quad (5)$$

for some permutation matrix $\mathbf{Q}$, where $\mathbf{E} = \mathbf{e}\mathbf{e}^T$ is the $(2K+1) \times (2K+1)$ matrix of all 1's. Two particularly interesting special cases are the following. First, taking $\mathbf{Q}$ as the identity matrix and setting β to zero yields the Hammerstein model [14]

$$y_k = \sum_{i=-K}^{K} \left(\frac{y_0}{2K+1} + Ax_{k-i} + \alpha x_{k-i}^2 \right), \quad (6)$$

and setting α to zero yields the Wiener model [14]

$$y_k = y_0 + A \left(\sum_{i=-K}^{K} x_{k-i} \right) + \beta \left(\sum_{i=-K}^{K} x_{k-i} \right)^2. \quad (7)$$

As a more general example, note that the following qaudratic filter also belongs to the class $\mathcal{C}_1$:

$$y_k = \alpha[x_{k-1}x_{k+1} + x_k x_{k-1} + x_k x_{k+1}] \quad (8)$$

obtained by choosing the permutation matrix

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \quad (9)$$

Like the median filter, note that the impulse response of this Volterra filter is identically zero, in contrast to both the Hammerstein and Wiener filters (6) and (7). In fact, it is not difficult to see that the impulse response of a nontrivial quadratic filter in $\mathcal{P}_K$ (i.e., any model of the form (4) with $y_0 = A = 0$) will be identically zero if and only if $\beta = 0$ in Eq. (5) and $\mathbf{Q}$ is a permutation matrix for which $Q_{0,0} \neq 1$.

3. THE FILTER CLASS $\mathcal{R}_K$

Generalizations of the L-filters defined in Eq. (2) that combine both temporal and rank ordering [2, 6, 13] do not belong to the class $\mathcal{P}_K$. Similarly, other extensions of the median filter like the FMH filters of Heinonen and Neuvo [9] that distinguish between the past elements x_{k-i} and the future elements x_{k+i} for $i > 0$ are also not included in $\mathcal{P}_K$. Conversely, symmetric versions of these filters do belong to the class $\mathcal{R}_K$, along with the classical linear symmetric smoothing filters [8]. Thus, an understanding of the consequences of the symmetry restriction defining $\mathcal{P}_K$ may provide useful insights into the behavioral differences between these different filter types. The class $\mathcal{R}_K$ also includes a wider variety of Volterra filters, but they remain highly structured. For example, the quadratic parameter matrix $\mathbf{B}$ defining a 2^{nd}-order Volterra filter is necessarily centrosymmetric ($\mathbf{JBJ} = \mathbf{B}$), a class that includes all symmetric Toeplitz matrices.

An observation that leads to definitions of new filters in class $\mathcal{R}_K$ is the following. First, define the K-vectors

$$\begin{aligned} \mathbf{w}_k^- &= [w_{k-K}, \ldots, w_{k-1}]^T \\ \mathbf{w}_k^+ &= [w_{k+1}, \ldots, w_{k+K}]^T \end{aligned} \quad (10)$$

and rewrite the filter output as $y_k = \Phi(\mathbf{w}_k^-, x_k, \mathbf{w}_k^+)$. Membership in class $\mathcal{R}_K$ therefore implies

$$\Phi(\mathbf{w}_k^-, x_k, \mathbf{w}_k^+) = \Phi(\tilde{\mathbf{w}}_k^+, x_k, \tilde{\mathbf{w}}_k^-) \quad (11)$$

where $\tilde{\mathbf{w}}_k^\pm$ denotes the time reversal of the subvector $\mathbf{w}_k^\pm$. This requirement is clearly satisfied if $\Phi(\cdot)$ is of the form

$$\Phi(\mathbf{w}_k^-, x_k, \mathbf{w}_k^+) = \Gamma[\phi(\mathbf{w}_k^-), x_k, \tilde{\phi}(\mathbf{w}_k^+)] \quad (12)$$

for arbitrary $\phi : R^K \to R$, where $\tilde{\phi}(\cdot)$ is the function $\phi(\cdot)$ with its arguments reversed and $\Gamma : R^3 \to R$ is any function satisfying

$$\Gamma(x, y, z) = \Gamma(z, y, x) \quad (13)$$

for all real x, y, and z. Taking ϕ as the unweighted average and Γ as the median yields the basic FMH filter, which may be regarded as a prototype for the class $\mathcal{R}_K$. Conversely, taking $\phi(\cdot)$ as an arbitrary Volterra model leads to a new variant of the FMH filter.

4. THE FILTER CLASS $\mathcal{C}_K$

The class $\mathcal{C}_K$ is quite different from either $\mathcal{P}_K$ or $\mathcal{R}_K$. In particular, note that filters in $\mathcal{C}_K$ all exhibit zero responses to constant inputs, and $\mathcal{C}_K$ includes both the classical linear antisymmetric differentiation filters [8], and some interesting nonlinear extensions. Specifically, note that the filter class $\mathcal{C}_K$ includes all filters satisfying Eq. (12) where the symmetry condition (13) is replaced by the antisymmetry condition

$$\Gamma(x, y, z) = -\Gamma(z, y, x) \quad (14)$$

for all real x, y, and z. Here, note the similarity to the general FMH structure discussed above: these $\mathcal{C}_K$ filters may be viewed as an antisymmetric combination of the forward prediction $\phi(\mathbf{w}_k^-)$ and the backward prediction $\tilde{\phi}(\mathbf{w}_k^+)$.

As a specific example, consider the nonlinear filter

$$\begin{aligned} y_k = \; & A[\text{med}\{x_{k+1}, ..., x_{k+K}\} \\ & - \text{med}\{x_{k-1}, ..., x_{k-K}\}]. \end{aligned} \quad (15)$$

If $K = 2q + 1$, the response of this filter to any monotonically increasing or decreasing sequence is the linear derivative approximation $y_k = A[x_{k+q+1} - x_{k-q-1}]$. For $q = 0$, this filter reduces to the linear central difference approximation for all input sequences, but for $q > 0$ the nonlinear differentiator is immune to isolated "spikes" in the data, in sharp contrast to extreme sensitivity of linear differentiation filters to these artifacts.

As another example of a nonlinear filter in $\mathcal{C}_K$, consider

$$y_k = |x_k - \frac{1}{K}\sum_{i=1}^{K} x_{k-i}| - |x_k - \frac{1}{K}\sum_{i=1}^{K} x_{k+i}|. \quad (16)$$

In contrast to the previous examples, this fillter illustrates that not all functions satisfying the antisymmetry condition (14) are independent of the central variable y. In fact, note that any function of the form

$$\Gamma(x,y,z) = \gamma(x,y) - \gamma(z,y) \quad (17)$$

satisfies this condition, for arbitrary $\gamma : R^2 \rightarrow R$. Similarly, products of an odd number of distinct functions satisfying Eq. (14) also satisfy this condition.

5. SYMMETRIES AND RELATIONS BETWEEN CLASSES

Given an arbitrary member of any of the three filter classes just described, it is possible to construct an uncountably infinite number of additional examples on the basis of the following results. First, if $f : R^n \rightarrow R^m$ and $g : R^m \rightarrow R^p$, the composition $g \circ f$ is defined as

$$g \circ f(x) = g(f(x)). \quad (18)$$

Note that if $\Phi \in \mathcal{R}_K$ or $\Phi \in \mathcal{P}_K$ then so is $f \circ \Phi$ for any scalar function f. Next, recall that a function f is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$. If $\Phi \in \mathcal{C}_K$ then $f \circ \Phi \in \mathcal{C}_K$ *provided* f is odd; conversely, if f is even it follows that $f \circ \Phi \in \mathcal{R}_K$.

These observations may be used to define *symmetric generalized Volterra filters*, as follows. Denote by $V_{(N,2K+1)}$ the following finite Volterra model [14, ch. 5]:

$$\begin{aligned} y_k &= y_0 + \sum_{n=1}^{N} v_n(k) \\ v_n(k) &= \sum_{i_1=-K}^{K} \cdots \sum_{i_n=-K}^{K} \alpha_n(i_1,\ldots,i_n) \\ &\quad \cdot x_{k-i_1} \cdots x_{k-i_n}. \end{aligned} \quad (19)$$

The corresponding generalized Volterra filter is defined as

$$y_k = y_0 + \sum_{n=1}^{N} f_n(v_n(k)) \quad (20)$$

where $f_n : R \rightarrow R$ is arbitrary. This filter will belong to class $\mathcal{P}_K$ or $\mathcal{R}_K$ if the Volterra coefficieints $\alpha_n(i_1,\ldots,i_n)$ satisfies the symmetry conditions noted in Secs. 2 or 3, respectively. Similar reasoning applies to the class $\mathcal{C}_K$ provided the scalar functions $f_n(\cdot)$ exhibit odd symmetry; again, if these functions exhibit even symmetry and the Volterra coefficients satisfy the requirements for inclusion in $\mathcal{C}_K$, the corresponding generalized Volterra filter will belong to the class $\mathcal{R}_K$.

A much more flexible extension of this idea is the following one. Define $\mathcal{N}$ to be any endomorphism of R^{2K+1} (i.e., any mapping of this vector space into itself), consider $\Phi \in \mathcal{P}_K$ and assume that $\mathcal{N} \circ \mathcal{P} = \mathcal{P} \circ \mathcal{N}$ for all permutations $\mathcal{P}$. It follows immediately that

$$\begin{aligned} \Phi \circ \mathcal{N} \circ \mathcal{P} &= \Phi \circ \mathcal{P} \circ \mathcal{N} = \Phi \circ \mathcal{N} \\ \Rightarrow \Phi \circ \mathcal{N} &\in \mathcal{P}_K. \end{aligned} \quad (21)$$

Similarly, if $\mathcal{N} \circ \mathbf{J} = \mathbf{J} \circ \mathcal{N}$ then $\Phi \circ \mathcal{N}$ belongs to $\mathcal{R}_K$ if Φ does. As a specific example, note that if $\mathcal{N}$ is of the form $\mathcal{N}[\mathbf{x}] = \text{diag}\{g(x_k)\}$, it follows that $\mathcal{N} \circ \mathcal{P}$ for any permutation $\mathcal{P}$. If we further assume the scalar function $g(\cdot)$ is invertible, the following composition defines the class of homomorphic systems [2, 14, 15]:

$$\Psi = g^{-1} \circ \Phi \circ \mathcal{N}. \quad (22)$$

Note that if Φ is a filter of class $\mathcal{P}_K$ or $\mathcal{R}_K$, the homomorphic filter also belongs to this class; again, if $g(\cdot)$ is an odd function and Φ belongs to class $\mathcal{C}_K$, the homomorphic filter also belongs to class $\mathcal{C}_K$. As a specific example, note that all uniformly weighted nonlinear mean filters [2, 15] belong to class $\mathcal{P}_K$ and all symmetrically weighted nonlinear mean filters belong to class $\mathcal{R}_K$. As a more unusual example, note that the homomorphic filters obtained from the median filter or any other L-filter also belong to class $\mathcal{P}_K$.

More generally, suppose $\mathcal{N}$ is an endomorphism of R^{2K+1} and denote each component of this mapping by $\eta_i : R^{2K+1} \rightarrow R$ for $i = -K,\ldots,K$. It follows immediately that if $\eta_i \in \mathcal{P}_K$ for all i then $\mathcal{N} \circ \mathcal{P} = \mathcal{N}$ for all permutations $\mathcal{P}$ and the composition $\Phi \circ \mathcal{N}$ belongs to $\mathcal{P}_K$ if Φ does. This observation is interesting in part because the rank-ordering operator satisfies this condition for all permutations $\mathcal{P}$. Similarly, if $\eta_i \in \mathcal{R}_K$ for all i, it follows that $\mathcal{N} \circ \mathbf{J} = \mathcal{N}$ and $\Phi \circ \mathcal{N}$ belongs to $\mathcal{R}_K$ if Φ does. Finally, a similar result applies if $\eta_i \in \mathcal{C}_K$ for all i, but again with restrictions: there, $\mathcal{N} \circ \mathbf{J} = -\mathcal{N}$, implying $\Phi \circ \mathcal{N}$ belongs to class $\mathcal{C}_K$ if Φ is an odd-symmetry mapping of class $\mathcal{R}_K$.

6. ROOT SEQUENCES

If a sequence $\{x_k\}$ is invariant to a filter Φ, it is called a root sequence and the characterization of root sequences is a topic of considerable interest in nonlinear digital filtering [1, 2, 3, 7, 12]. The following discussion briefly considers the influence of the symmetry classes introduced here on the character of these root sequences.

First, consider the class $\mathcal{P}_K$ and note that the response to constant sequences is constant; hence, the constant sequence $x_k = c$ will be a root for the filter $\Phi \in \mathcal{P}_K$ if and only if $\Phi(c, \ldots, c) = c$. Note that sufficient conditions for this result to hold are Mallows' criteria A2 (location invariance, $\Phi(\mathbf{w}_k + \mathbf{c}) = \Phi(\mathbf{w}_k) + c$) and A3 (the filter is centered, $\Phi(\mathbf{0}) = 0$) [11]. To search for more interesting root sequences, consider the class of periodic sequences with period P. For $P = 2$, the sequence $\{x_k\}$ may be written as $\ldots ababa \ldots$, and the the requirement for a root sequence reduces to

$$\Phi(a, b, a) = b \qquad \Phi(b, a, b) = b. \tag{23}$$

For $K = 1$, this condition cannot be satisfied by smoothing filters like the median filter or the unweighted average, but it can be satisfied by filters that extract extreme values. For example, define μ as the median of the values x, y, and z, and consider the filter defined by the function

$$\Phi(x, y, z) = \begin{cases} x & |x - \mu| > |y - \mu|, |z - \mu| \\ y & |y - \mu| \geq |x - \mu|, |z - \mu| \\ z & |z - \mu| > |x - \mu|, |y - \mu|. \end{cases} \tag{24}$$

Note that any period 2 sequence satisfies Eq. (23) for this filter. Conversely, for $K > 1$, period 2 roots are more common. In particular, it is known that arbitrary binary sequences are roots of median filters of even half-width K [1].

Similarly, any period 3 sequence may be expressed as

$$\ldots, \quad a, b, c, a, b, c, \quad \ldots \tag{25}$$

and if it is a root sequence, it must satisfy the conditions

$$\begin{aligned} \Phi(a, b, c) &= b \\ \Phi(b, c, a) &= c = \Phi(a, b, c) = b \\ \Phi(c, a, b) &= a = \Phi(a, b, c) = b. \end{aligned} \tag{26}$$

In other words, any period 3 root sequence is necessarily constant for a filter in class $\mathcal{P}_1$. More generally, the same construction shows that any period $2K + 1$ root sequence is necessarily constant for any filter in class $\mathcal{P}_K$ since any data window contains precisely one period of the sequence, differing only in order. The same argument extends to periodic roots of period $P = (2K + 1)/n$ for any integer n since the data window then includes n complete periods of the sequence for all k.

A similar argument may also be used to show that impulses cannot be root sequences of any filter in $\mathcal{P}_K$. Specifically, it follows by permutation symmetry that the filter response is necessarily constant for any window that contains the nonzero value of the impulse. Hence, the impulse response of any filter in class $\mathcal{P}_K$ is of width $2K + 1$; note that this response can be identically zero, as in the case of the median filter, but any $\mathcal{P}_K$ filter with a nonzero impulse response exhibits the same character as the linear moving average filter.

Since $\mathcal{P}_K \subset \mathcal{R}_K$, any type of root sequence that is possible for class $\mathcal{P}_K$ is also possible for class $\mathcal{R}_K$. Further, note that the identity filter $y_k = x_k$ belongs to the class $\mathcal{R}_K$ for all K, so this symmetry requirement imposes no restrictions on the possible root sequences: *any* sequence can be a root for some filter in $\mathcal{R}_K$. Conversely, the symmetry condition defining the class $\mathcal{C}_K$ imposes some rather stringent restrictions on the character of the possible root sequences. For example, since the response of any filter in this class to constant sequences is identically zero, it follows that the only constant root sequence is the zero sequence. In fact, it is easily demonstrated that this conclusion extends to period 2 sequences for any class $\mathcal{C}_K$. This point is most easily seen for the case $K = 1$, but the result extends readily to arbitrary K. Specifically, for a period 2 sequence to be a root sequence, it is necessary that

$$\begin{aligned} \Phi(a, b, a) &= b = -\Phi(a, b, a) \Rightarrow b = 0 \\ \Phi(b, a, b) &= a = -\Phi(b, a, b) \Rightarrow a = 0. \end{aligned} \tag{27}$$

In contrast, it is possible for filters in the class $\mathcal{C}_K$ to exhibit nontrivial periodic root sequences of period $2K + 1$. This point is illustrated for $K = 1$ by the following example. Consider the period 3 sequence (25) and suppose it is a root, implying

$$\Phi(a, b, c) = b \quad \Phi(c, a, b) = a \quad \Phi(b, c, a) = c. \tag{28}$$

Values for $\Phi(c, b, a)$, $\Phi(b, a, c)$, and $\Phi(a, c, b)$ follow from these conditions by symmetry, but these conditions together only define 6 of the 27 values for the function $\Phi(x, y, z)$ when the variables x, y, and z are restricted to the values a, b, and c. Of the remaining 21 values, 9 are identically zero by the antisymmetry restrictions; six of the remaining 12 function values may be specified arbitrarily and the other six are determined from these six by the antisymmetry restriction. Hence, even if we restrict consideration to endomorphisms of the discrete set of values $\{a, b, c\}$ (implyint that one of these values is zero), we are left with $6^3 = 216$ consistent ways of specifying the remaining function values.

Next, consider the question of binary root sequences for the filter class $\mathcal{C}_K$. It follows from the antisymmetry requirement defining $\mathcal{C}_K$ that any nontrivial binary root sequence must assume the values $\pm a$ for some $a \neq 0$. Further, since $y_k = 0$ identically for constant sequences, it follows that binary root sequences cannot contain subsequences of length $2K + 1$. Similarly, if a root sequence contained a subsequence of length $2K$, it would also have to include the subsequence

$$\ldots, \quad -a, a, \ldots, a, -a, \quad \ldots$$

to avoid subsequences of length $2K + 1$. However, the antisymmetry requirement then leads to a contradiction:

$$\Phi(-a, a, \ldots, a, a) = -\Phi(a, a, \ldots, a, -a) \quad \Rightarrow \quad a = -a.$$

Hence, constant subsequences of length $2K$ are also forbidden in a binary root sequence. Similarly, subsequences of length $2K-1$ are also inadmissible in root sequences because they would have to be imbedded in a symmetric sequence of length $2K+1$ to avoid the creation of subsequences of length $2K$ or $2K+1$, thus implying $y_k = 0$. As an immediate corollary, it follows that the filters in class $\mathcal{C}_1$ cannot exhibit nontrivial binary root sequences. Conversely, binary root sequences *are* possible for $K > 1$, as the following example illustrates. Specifically, the period 6 sequence based on the subsequence $a, a, -a, -a, a, -a$ can be a root sequence for a filter in class $\mathcal{C}_2$ that maps bilinear sequences into the set $\{-a, 0, a\}$.

Finally, as an interesting application of these root sequence results, suppose $\Phi \in \mathcal{C}_K$ is of the form (12) for $K = 2q+1$ and $\phi \in \mathcal{R}_q$. It follows that $\tilde{\phi}(\mathbf{w}_k^+) = \phi(\mathbf{w}_k^+)$ and $y_k = \Gamma(\phi(\mathbf{w}_k^-), x_k, \phi(\mathbf{w}_k^+))$. Further, if $\{x_k\}$ is a root sequence for $\phi(\cdot)$, then $\phi(\mathbf{w}_k^-) = x_{k-q-1}$ and $\phi(\mathbf{w}_k^+) = x_{k+q+1}$, implying

$$y_k = \Gamma(x_{k-q-1}, x_k, x_{k+q+1}). \tag{29}$$

The response of the median differentiator defined in Eq. (15) to monotonic sequences follows directly from this result.

7. SUMMARY

This paper has defined three classes of symmetric nonlinear moving average filters, $\mathcal{P}_K$, $\mathcal{R}_K$, and $\mathcal{C}_K$, all of which include popular filter classes as subsets; in addition, new members of each class have been described here, along with some characterizations of filters within these classes and systematic procedures for constructing additional filters in each class. For example, a complete characterization of quadratic filters in class $\mathcal{P}_K$ was given, including necessary and sufficient conditions for impulse rejection; in addition, the class of generalized Volterra models was defined and conditions for these models to belong to each of these symmetry classes were given. In general, the FMH filter appears to be the prototype for filters in the class $\mathcal{R}_K$, while the class $\mathcal{C}_K$ appear to be generalized differentiation filters. The class $\mathcal{P}_K$ is required to be invariant under the permutation group defined by the data window whereas the class $\mathcal{R}_K$ is only required to be invariant with respect to the permutation subgroup defined by time reversal. The defining conditions for $\mathcal{P}_K$ are restrictive enough to exclude nonconstant period $2K+1$ sequences and impulses from the root set, but the symmetry conditions defining the class $\mathcal{R}_K$ do not impose any restrictions on possible root sequences for these filters. An interesting extension would be to consider other subgroups of the permutation group, exploring both the resulting filter structures and their root sequences.

8. REFERENCES

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