NONSTATIONARY TEXTURE MODELLING BY $\mathcal{G}$-INVARIANT RANDOM FIELDS

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ABSTRACT

In image processing, textures are generally represented as homogeneous random fields, homogeneous meaning stationary or second-order stationary. This paper presents a generalization of the second-order stationarity to the second-order invariance under a group of transforms. Some examples of interesting groups are given. The Cholesky factorization is applied for the synthesis of random fields showing this generalized invariance property.

1. INTRODUCTION

Random field modelling of textures is based on some invariance property in the image, and the translation invariance of statistical features like covariance is commonly used, which leads to the classical form of stationarity and of standard textures. The generalization of this concept leads to statistical invariance (also called homogeneity) of a random field under a group of transformations in order to model nonstandard textures. The general formulation includes the group of translations in the plane as a particular case.

An early work on this subject by Yaglom can be found in [1] where the spectral decomposition for second-order homogeneous random fields has been studied, the homogeneity being defined with respect to group structure. The concept of invariance of a random field under a general group composition law has also been considered by Hannan [2] from a theoretical probabilistic point of view. However, these general studies were not intended for prctical applications. Concerning 1D signals, the special case of multiplicative-invariant processes has raised some interest [3, 4, 5]. Concerning 2D signals, special attention to the applications of the group theory to image processing has been paid by Lenz [6]. In the texture modelling field, we have recently proposed [7] a model for second-order scale-invariant random fields.

In this paper, we propose a general formulation for random fields that are second-order invariant with respect to a group of transformations in the plane.

2. $\mathcal{G}$-INVARIANT RANDOM FIELDS

A standard assumption in homogeneous image modelling is texture stationarity, which can be interpreted as stating the invariance of some probabilistic features, namely the mean value and the covariance functions, with respect to some family of transformations in the plane, namely the translations in the standard case. In this paper, we restrict ourselves to the study of second-order property invariance (mean value and covariance function) with respect to groups of transformations operating on $\mathcal{D}$, a subset of $\mathbb{R}^2$. The group $\mathcal{G}$ of transformations operating on $\mathcal{D}$ is the group of all bijective maps of $\mathcal{D}$.

A complex valued random field X, defined on $\mathcal{D}$, is said to be second-order $\mathcal{G}$–stationary if and only if :

$$E[X(g(u))] = E[X(u)] \tag{1}$$

$$E[X(g(u)).\overline{X(g(v))}] = E[X(u).\overline{X(v)}] \tag{2}$$

$$(\forall g \in \mathcal{G}, \forall (u,v) \in \mathcal{D}^2)$$

In the classical case, when $\mathcal{G}$ is the group of all translations operating on $\mathcal{D} = \mathbb{R}^2$, equation (1) implies that the mean value function $m(u) = E[X(u)]$ is constant on $\mathcal{D}$. In the general case, equation (1) only implies that $m(u) = m(v)$ as soon as there exists some g in $\mathcal{G}$ such that $v = g(u)$. In the sequel we shall restrict ourselves to the case of centered processes.

3. $\mathcal{G}$-ORBITS AND $\mathcal{G}$-INVARIANTS

3.1. Basic notions

Group theory concepts come up with a convenient interpretation of the relation satisfied by the covariance function $C(u,v) = E[X(u).\overline{X(v)}]$ of any $\mathcal{G}$–invariant random field, the covariance of which can be shown to satisfy :

$$C(u,v) = C(g(u), g(v)) = C(g(u,v)) \tag{3}$$

Two couples (u_1, v_1) and (u_2, v_2) are said to be $\mathcal{G}$–equivalent ($(u_1, v_1) \sim (u_2, v_2)$) if there exists some $g \in \mathcal{G}$ such that $(u_2, v_2) = g(u_1, v_1)$. This is a true equivalence relation, the equivalence classes of which are called $\mathcal{G}$–orbits. Thus, relation (3) implies that the covariance function is constant on each $\mathcal{G}$–orbit.

The former abstract notion of orbit set can be turned into a practical one by introducing the concept of invariants as follows. A function $i(u,v)$, defined on $\mathcal{D}^2$ and taking its values in some set I, is said to be $\mathcal{G}$–invariant if :

$$(u_1, v_1) \sim (u_2, v_2) \Rightarrow i(u_1, v_1) = i(u_2, v_2) \tag{4}$$

Furthermore, i is said to be maximal $\mathcal{G}$–invariant if :

$$(u_1, v_1) \sim (u_2, v_2) \Leftrightarrow i(u_1, v_1) = i(u_2, v_2) \tag{5}$$

It is a standard result of invariant function theory [8] that any $\mathcal{G}$–invariant is a function of any maximal $\mathcal{G}$–invariant. The notion of invariants leads to practical and efficient tools, which will be presented and discussed through several examples in the sequel.

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3.2. Examples

Usual invariants take their values in $I = \mathbb{R}^q, q \in \mathbb{N}$; as $\mathcal{D}^2$ itself is a subset of $\mathbb{R}^4$, the only cases of interest are $q = 1, 2$ and 3. Some illustrations will be presented in the following examples with $\mathcal{D} = \mathbb{R}^2$. In these examples, and in the sequel as well, u and v are pixel sites $u = (x, y), v = (z, t)$ or vectors $u = \begin{pmatrix} x \\ y \end{pmatrix}, v = \begin{pmatrix} z \\ t \end{pmatrix}$, or complex numbers $u = x + iy, v = z + it$, the interpretation being clear from the context.

Example 1

$\mathcal{G}_1$ is the group of all translations in the plane ($g(u) = u + h, h \in \mathcal{D}$). The following relations are equivalent to relation $(u_1, v_1) \sim (u_2, v_2)$:

$$\exists\, h \text{ such that } u_2 = u_1 + h \text{ and } v_2 = v_1 + h \tag{6}$$

$$u_2 - u_1 = v_2 - v_1 \tag{7}$$

$$u_2 - v_2 = u_1 - v_1 \tag{8}$$

Thus $i_1(u, v) = (u - v)$ is a maximal $\mathcal{G}_1$–invariant, $q = 2$, and :

$$C(u, v) = \Gamma(u - v) \tag{9}$$

which corresponds to the classical second order stationarity, with the usual properties (among them, Hermitian symmetry and pistive-definiteness).

Example 2

$\mathcal{G}_2$ is the group of Euclidean motions, the group of all rotations about any center in the plane ($g(u) = Ru + h, h \in \mathcal{D}, R \in SO(2)$). The following relations are equivalent to relation $(u_1, v_1) \sim (u_2, v_2)$:

$$\exists\, h \text{ and } R \text{ such that } u_2 = Ru_1 + h \text{ and } v_2 = Rv_1 + h \tag{10}$$

$$\exists\, R \text{ such that } u_2 - Ru_1 = v_2 - Rv_1 \tag{11}$$

$$\exists\, R \text{ such that } u_2 - v_2 = R(u_1 - v_1) \tag{12}$$

$$||u_2 - v_2|| = ||u_1 - v_1|| \tag{13}$$

the last relation holding since there always exists a rotation transforming any vector into any other vector having the same Euclidean norm. Thus, $i_2(u, v) = ||u-v||$ is a maximal $\mathcal{G}_2$–invariant, $q = 1$, and :

$$C(u, v) = \Gamma\Big(||u - v||\Big) \tag{14}$$

which corresponds to the well known case of isotropy.

Example 3

Simple similar considerations leads for $\mathcal{G}_3$, the group of linear isometries ($g(u) = Su, S \in O(2)$), to show that $C(u, v) = \Gamma\Big(||u||, < u, v >, ||v||\Big)$.

Example 4

It can be shown in a similar way that for $\mathcal{G}_4$, the group of all dilations with strictly positive ratio ($g(u) = \lambda u + b$ in complex notation, $\lambda > 0, b \in \mathbb{C}$), $C(u, v) = \Gamma(\text{ arg } (u - v))$

4. GENERALIZED TRANSLATION APPROACH

4.1. Introduction

Assume that $\mathcal{D}$ itself is a group under some composition law, written multiplicatively, as in classical stationarity, where the plane is a group under vector translation. Let $\mathcal{G}$ be the group of all (generalized) translations under the composition law $\mathcal{G} = \{T_u, u \in \mathcal{D}\}$, where T_u is defined by :

$$T_u(v) = u.v \qquad \forall v \in \mathcal{D} \tag{15}$$

Thus, the following proposition can be formulated (demonstration is straightforward) :

Proposition (1) Assume that $\mathcal{G}$ is the group of all translations operating on group $\mathcal{D}$. For a positive-definite kernel C, defined on $\mathcal{D}^2$, to be the covariance function of some $\mathcal{G}$–stationary (centered) random field on $\mathcal{D}$, it is necessary and sufficient that there exists some complex-valued function Γ, defined on $\mathcal{D}$, such that :

$$C(u, v) = \Gamma\left(v^{-1}.u\right) \tag{16}$$

4.2. Examples

Example 5

$\mathcal{G}_5$, the group of all positive scale changes defined on $\mathcal{D} = \{u = (x, y), x > 0, y > 0\}$ by $g_{a,b}(u) = (ax, by), a > 0, b > 0$, is an example of application of proposition 1. It suffices to put

$$\text{for } u = (x, y) \text{ and } v = (z, t),\ u.v = (xz, yt) \tag{17}$$

to make $\mathcal{D}$ a group and derive

$$C(u, v) = C\Big((x, y), (z, t)\Big) = \Gamma\left(\frac{x}{z}, \frac{y}{t}\right) \tag{18}$$

from (16).

Example 6

$\mathcal{G}_6$ is the group of all similarities about O. Adopting the complex notation, the similarity about O with positive ratio λ and angle α is viewed as multiplying u by $z = \lambda e^{i\alpha}$ or being the (generalized) translation T_z defined by $T_z(u) = zu$ on $\mathcal{D} = \mathbb{C} - \{O\}$. (16) is written as :

$$C(u, v) = \Gamma\left(\frac{u}{v}\right) \text{ or, in real notation,} \tag{19}$$

$$C(u, v) = \Gamma\left(\frac{|u|}{|v|}, \text{ arg } u - \text{ arg } v\right) \tag{20}$$

where $|z|$ is the modulus of z.

$\mathcal{G}_6$–stationarity can be shown to be connected with self-similarity.

4.3. An important special case

An important special case is related to the situation where $\mathcal{G}$, the group of all generalized translations on $\mathcal{D}$, is isomorphic to some subgroup of the group of standard translations in the plane. In such a case, a practical way arises to produce $\mathcal{G}$–stationary covariance functions from the knowledge of classical stationary functions. Suppose that $(\mathcal{D}, \cdot)$ is isomorphic to $(\mathcal{D}_0, +)$, by isomorphism φ, the inverse of which is ψ. Let $\mathcal{G}$ be the group of all generalized translations on $(\mathcal{D}, \cdot)$. Random field X defined on $\mathcal{D}$ and random field Y defined on $\mathcal{D}_0$ verify :

$$Y(s) = X(\psi(s)) \text{ or, equivalently, } X(u) = Y(\varphi(u)) \tag{21}$$

Thus, X is $\mathcal{G}$–stationary $\iff Y$ is stationary in the classical sense. Moreover, if $C_X(u,v) = \Gamma_X(uv^{-1})$ and $C_Y(s,t) = \Gamma(s-t)$, Γ_X and Γ_Y are related by :

$$\Gamma_Y(h) = \Gamma_X(\psi(h)), h \in \mathcal{D}_0 \text{ or } \Gamma_X(w) = \Gamma_Y(\varphi(w)), w \in \mathcal{D} \quad (22)$$

Examples

Many of the previous examples can be dealt with using this approach, provided a suitable isomorphism is defined. Suppose, in the sequel of this section, that Γ, a complex-valued function defined on $\mathcal{D}_0$, is associated with the covariance function of a classical stationary process by $C(u,v) = \Gamma(u-v)$.

The following typical examples (where $h = (h_1, h_2)$) will be useful further for simulation of random fields :

- Separable case

$$\Gamma_1(h) = \sigma^2 \rho_1^{|h_1|} \rho_2^{|h_2|}, \qquad 0 < \rho_i < 1, i = 1,2 \quad (23)$$

- Isotropic case

$$\Gamma_2(h) = \sigma^2 \rho^{||h||}, \qquad 0 < \rho < 1 \quad (24)$$

- Cyclic case

$$\Gamma_3(h) = \sigma^2 \cos h_1 \cos h_2 \quad (25)$$

- Self-similarity case

$$\Gamma_4(h) = \frac{\sigma^2}{2}\left[\mathrm{e}^{Hh_1} + \mathrm{e}^{-Hh_1} + \left(\mathrm{e}^{h_1} + \mathrm{e}^{-h_1} - 2\cos h_2 \right)^H \right] \quad (26)$$

A random field X will be $\mathcal{G}$–stationary as soon as :

$$C_X(u,v) = \Gamma(\varphi(u) - \varphi(v)) = \Gamma(\varphi(uv^{-1})) \quad (27)$$

Example 7 (Perspective planar transform)

Consider the perspective projection of an object surface S onto an image plane I, at focal distance f from the camera plane, where (O,X,Y) is the camera coordinate system and (O,x,y) the image coordinate system. The image plane is defined by $Z = f$. In Computer Vision, a perspective projection p is defined by the focal length f, the distance δ, the slant σ and the tilt τ. The slant is the angle between the optic axis and the surface normal. The tilt is the angle between the parallel projection of the surface normal onto the image plane and the x axis. δ is the perpendicular distance to the plane from the origin O_{XYZ} and f is the focal length of the camera.

The projection, p, relates the coordinates $v = (z,t)$ of a point in the object plane to the coordinates $u = (x,y)$ of a point in the image plane by $u = p(v) = (p_1(v), p_2(v))$. Conversely, the backprojection is $p^{-1}(u) = (p_1^{-1}(u), p_2^{-1}(u))$.

$$x = p_1(v) = \frac{f}{\sin\sigma z + \delta\cos\sigma}(\cos\sigma\cos\tau z - \sin\tau t - \delta\sin\sigma\sin\tau) \quad (28)$$

$$y = p_2(v) = \frac{f}{\sin\sigma z + \delta\cos\sigma}(\cos\sigma\sin\tau z + \cos\tau t - \delta\sin\sigma\sin\tau) \quad (29)$$

The transform p is a projective transform if its determinant satisfies:

$$\begin{vmatrix} \cos\sigma\cos\tau & -\sin\tau & -\delta\sin\sigma\cos\tau \\ \cos\sigma\sin\tau & \cos\tau & -\delta\sin\sigma\sin\tau \\ \sin\sigma & 0 & \delta\cos\sigma \end{vmatrix} \neq 0 \quad (30)$$

The group of transforms considered $\mathcal{G}_7$ ($g_7(u), u \in \mathcal{D}$) is defined by:

$$g_7(u) = p \circ g_1 \circ p(u) \quad (31)$$

where $g_1(v) = v + h$ is a translation on $\mathcal{D}$.

The idea behind this construction is to represent in the plane the $\mathcal{G}$-invariance of a random field which is a projection of a $\mathcal{G}_1$ stationary field in the object plane S. The group $\mathcal{G}_7$ is isomorphic to the group of translation.

Let us define, for $u \in \mathbb{R}^2$, $\varphi(u) = (p_1^{-1}(u), p_2^{-1}(u))$, where $p_1^{-1}(u)$ and $p_2^{-1}(u)$ are the backprojection rational functions. Thus X is $\mathcal{G}_7$-stationary if and only if:

$$C(u,v) = \Gamma\left(p_1^{-1}(u) - p_1^{-1}(v), p_2^{-1}(u) - p_2^{-1}(v)\right) \quad (32)$$

5. SIMULATION RESULTS

Texture synthesis can be performed in all the previous cases using the Cholesky factorization method [9]. Let $X \in \mathbb{R}^{M^2}$ be the random vector corresponding to an arbitrary 1D scan of the points of the $M \times M$ domain, of mean vector m and covariance matrix R. Without any loss of generality, we can assume the mean to be constant and equal to zero. Given that it is a covariance matrix, R is symmetric and positive. If R is definite positive, it can be factorized as :

$$R = LL^t \quad (33)$$

where L is a lower triangular matrix, which is known as the Cholesky factorization. Let $Y = L^{-1}X$. Using (33), it is straightforward to obtain :

$$E[YY^t] = E[L^{-1}XX^tL^{-1^t}] = I \quad (34)$$

i.e. the random vector Y is a white noise.

The synthesis method follows the previous steps : find L, and simulate a Gaussian random noise vector $Y \in \mathbb{R}^{M^2}$. The random field desired is then given by :

$$X = LY \quad (35)$$

On the one hand, the main advantage of this method is the fact that the second order moments of the simulated random vector X are exactly the theoretical ones. On the other hand, its main disadvantage is the heavy computational load and storage requirements necessary to perform the Cholesky factorization of a $M^2 \times M^2$ matrix. Practically, the simulations were limited to 64×64 pixels texture fields.

The theoretical correlation function is needed for the simulations, and we have used the functions $(\Gamma_1, \Gamma_2, \Gamma_3)$ given in the previous section by (23), (24) and (25). Figure 1 and Figure 2 show a few examples of texture fields simulated using the method described above. For all the groups of transforms considered, the invariants that describe the correlation contain two parameters : $i(u,v) = (\varphi(v) - \varphi(u)) = (h_1, h_2)$.

For all the simulations the variance of the model was $\sigma = 1$. All the fields shown in Figure 1 and Figure 2 were stretched to fit the dynamic range of possible gray levels [0, 255]. For the $\mathcal{G}_5$ and the $\mathcal{G}_6$ cases, the coordinate zero was replaced by $\epsilon = 0.1$ for the numerical implementaion, although, theoretically, ϵ can be as close to zero as possible.

For the simulations of the perspective projection, $f = \delta = M$, the tilt was set to $\tau\pi/2$, which means that there is no rotation between the image coordinate system and the projected surface coordinate system, and the slant to 2.0 rd.

The simulated fields shown exhibit characteristic visual properties related to the type of second-order invariance. In particular, if a transformation of the considered group is applied, the aspect of the simulated field is invariant. Except for the case where $\mathcal{G}$ is the group of translations in the plane, the random fields depend on the placement of the centre of the coordinate system, as well as on the position of the projected plane in the space for the case $\mathcal{G}_7$.

The $\mathcal{G}$-invariant fields exhibit anisotropic characteristics and strong directionalities. These characteristics make such models suitable for the synthesis of anisotropic flow like textures, studied by Rao [10, 11]. In his work, Rao models anisotropic textures as a superposition of two fields : a directional or flow field, which contains the visible geometric properties of the fields, and a textural second order stationary field. The model proposed above can be useful for a simultaneous representation of these two aspects, even though the formal link between the directional field observed and the geometric properties of the second-order invariance has not been clearly established yet.

6. CONCLUSION

In this paper, a general formulation of the random fields second-order invariant for a group of transforms is proposed and discussed in the light of some group theory concepts. In particular, it is shown that the correlation function of a second-order $\mathcal{G}$-invariant random field on a space $\mathcal{D}$ can be formally defined on the orbits of $\mathcal{D} \times \mathcal{D}$ for the group $\mathcal{G}$. The correlation function can be therefore characterized through the orbit invariant functions. The classical examples of stationarity and isotropy fit exactly in this formulation. The hypothesis that generalize the classical stationarity are outlined, and the particular case of a group of transforms in the plane isomorphic to the group of translations is detailed. In this particular case, the valid correlation functions for a $\mathcal{G}$-invariant field are directly deduced from the valid correlation function for the second-order stationary case.

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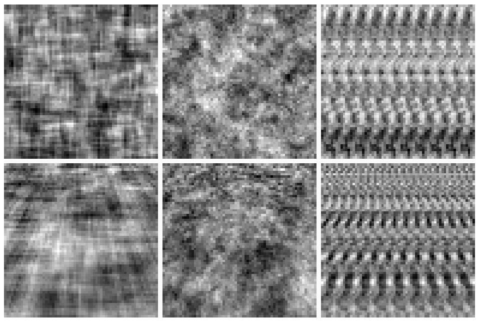

Figure 1: First row : standard textures based on translation invariance, from left to right a) $\Gamma_1(\rho_1 = \rho_2 = 0.9)$; b) $\Gamma_2(\rho = 0.7)$; c) $\Gamma_4(H = 0.7)$; second row : generalized textures based on $\mathcal{G}_7$-invariance, from left to right d) $\Gamma_1(\rho_1 = \rho_2 = 0.9)$; e) $\Gamma_2(\rho = 0.7)$; f) $\Gamma_4(H = 0.7)$.

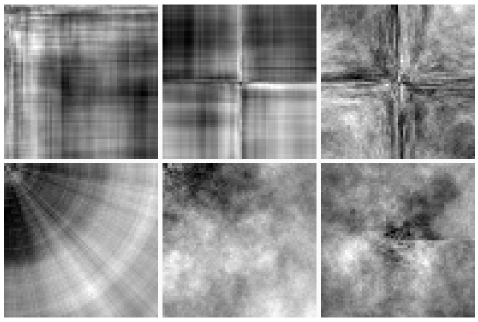

Figure 2: From top to bottom and from left to right : first row : generalized textures based on $\mathcal{G}_5$-invariance, a) $\Gamma_1(\rho_1 = \rho_2 = 0.9)$, center located at top left corner ; b) $\Gamma_2(\rho = 0.7)$, center located at $(x_c = 32, y_c = 32)$; c) $\Gamma_1(\rho_1 = \rho_2 = 0.7)$, center located at $(x_c = 32, y_c = 32)$; second row : generalized textures based on $\mathcal{G}_6$-invariance, d) $\Gamma_1(\rho_1 = \rho_2 = 0.9)$, center located at top left corner ; e) $\Gamma_2(\rho = 0.7)$, center located at $(x_c = 32, y_c = 32)$; f) $\Gamma_1(\rho_1 = \rho_2 = 0.7)$, center located at $(x_c = 32, y_c = 32)$.

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