NONLINEAR METHODS FOR THE IDENTIFICATION OF 2D MODELS WITH LONG-RANGE DEPENDENCE

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ABSTRACT

In this paper we propose an anisotropic model for 2D fractional noise generated from a Gaussian mixture distribution. We propose two identification methods for this model: the first one is a forward modeling approach based on even order cumulants of the field, while the second one is an inverse modeling approach where the synthesis filter is inverted in order to obtain an estimation of the driving noise. Simulation results are presented which show the effectiveness and the robustness of the proposed methods.

1. INTRODUCTION

In the last years, there has been an increasing interest in the characterization of images by "1/f" models (which present long-range dependence). In particular, several works have studied the use of the fractional Brownian motion (fBm) [3] for texture modeling [4]. These studies are, however, restricted to the isotropic case. As will be discussed in this paper, dual discrete-space models [1] may be more flexible and more tractable when dealing with real data. More precisely, we introduce a new model of "1/f" type for discrete textures, which takes into account the anisotropy often existing in natural scenes. This model can be seen as a twoJean-Christophe Pesquet

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dimensional extension of fractional ARIMA models [1, 2]. It is thus an anisotropic alternative to the 2D fractional Gaussian noise. Furthermore, the proposed model has an arbitrary probability distribution, approximated by a Gaussian mixture, which allows us to characterize a wide class of images. We propose a two-step nonlinear method to identify this model. The efficiency of the proposed method is illustrated by simulation examples.

The outline of this paper is as follows: in the next section we present the proposed model of anisotropic noise and derive some of its theoretical properties. Section 3 describes two identification methods for the parameters of this model. Section 4 presents the experimental results and allows us to draw some conclusions.

2. THE MODEL OF ANISOTROPIC FRACTIONAL NOISE

Our aim is to propose an anisotropic non-Gaussian discrete-space generalization of the fractional Gaussian noise (fGn). In 1D, the fGn is defined as a generalized random process whose "power spectrum density" is given by:

$$\mathcal{S}(\omega) \propto \frac{1}{|\omega|^{2d}}, \quad 0 < d < \frac{1}{2}.$$
 (1)

The dual discrete-time model of the fGn is an FARIMA(0, d, 0) with $d \in \mathbb{R}_+$ [1], denoted by u(n), such that:

$$\left(1-q^{-1}\right)^d u(n) = w(n)$$

where w(n) is a zero-mean white Gaussian noise and $q^{\pm 1}u(n) = u(n \pm 1)$. As $1 - q^{-1}$ is a discretetime derivation operator, $(1 - q^{-1})^d$ represents a discrete-time fractional derivation of order $d \in (0, 1/2)$. As $\omega \to 0$, the power spectrum density of the FARIMA(0,d,0) process tends to that of the fGn defined in (1).

A first isotropic 2D extension of the fGn is a generalized random field whose "power spectrum density" has the form:

$$\mathcal{S}(\omega_x, \omega_y) \propto rac{1}{\left(\omega_x^2 + \omega_y^2
ight)^{2d}}, \quad 0 < d < rac{1}{2}.$$
 (2)

Its discrete equivalent is an isotropic FARIMA which can be formally defined as:

$$\left[\left(1 - q_x^{-1} \right) \left(1 - q_x \right) + \left(1 - q_y^{-1} \right) \left(1 - q_y \right) \right]^d$$
$$u(n, m) = w(n, m)$$

where w(n, m) is a 2D zero-mean white Gaussian noise of variance σ_w^2 and $q_x^{\pm 1}$ (resp. $q_y^{\pm 1}$) denotes the shift operator in the x (resp. y) directions: $q_x^{\pm 1}u(n,m) = u(n \pm 1,m)$ (resp. $q_y^{\pm 1}u(n,m) = u(n,m \pm 1)$).

The power spectrum density of u(n,m) reads

$$\mathcal{S}(\omega_x, \omega_y) = \frac{\sigma_w^2}{4^{2d} \left(\sin^2 \frac{\omega_x}{2} + \sin^2 \frac{\omega_y}{2}\right)^{2d}}$$

In this case, as $\omega_x \to 0$ and $\omega_y \to 0$, the power spectrum density tends to that of the isotropic continuous-space model defined by (2).

In practice, it is often useful to have the ability of modeling anisotropic textures. In order to define such an anisotropic model, we propose an extension of the previous model by means of the following equation:

$$D^{d}_{\alpha,\varphi}\left(q_{x},q_{y}\right)u(n,m)=w(n,m). \tag{3}$$

The chosen operator $D_{\alpha,\varphi}^d(q_x,q_y)$ realizes a discrete fractional derivation of order d. It also depends on two additional parameters α and φ , which describe respectively the anisotropy in the image and its directionality (privileged orientation in the field). In our model, we also allow the driving noise w(n,m) to have a zero-mean non-Gaussian distribution, here approximated by a Gaussian mixture.

The form chosen for the operator $D^{d}_{\alpha,\varphi}\left(q_{x},q_{y}\right)$ corresponds to

$$D_{\alpha,\varphi}^{d} \left(e^{j\omega_{x}}, e^{j\omega_{y}} \right)$$

= $\left(\sin^{2} \frac{\omega_{x}}{2} + \sin^{2} \frac{\omega_{y}}{2} \right)^{d-1} \left[\sin^{2} \frac{\omega_{x}}{2} (1 - \alpha \cos \varphi) + \sin^{2} \frac{\omega_{y}}{2} (1 + \alpha \cos \varphi) - \frac{1}{2} \alpha \sin \varphi \sin \omega_{x} \sin \omega_{y} \right]$

The power spectrum density of u(n,m) is therefore given by

$$\mathcal{S}(\omega_x, \omega_y) = \frac{\sigma_w^2}{I(\omega_x, \omega_y)^{2(d-1)} A_{\alpha, \varphi}(\omega_x, \omega_y)^2} \quad (4)$$

where

$$I(\omega_x, \omega_y) = \sin^2 \frac{\omega_x}{2} + \sin^2 \frac{\omega_y}{2}$$

is an isotropic part only depending on the fractional derivation order d and

$$\begin{aligned} A_{\alpha,\varphi}(\omega_x,\omega_y) \\ &= (1 - \alpha\cos\varphi)\sin^2\frac{\omega_x}{2} \\ &+ (1 + \alpha\cos\varphi)\sin^2\frac{\omega_y}{2} - \frac{1}{2}\alpha\sin\varphi\sin\omega_x\sin\omega_y \end{aligned}$$

is an anisotropic part, depending on the parameters α and φ . From this expression, it is obvious that we can obtain all possible forms for $A_{\alpha,\varphi}(\omega_x,\omega_y)$ by considering $\alpha \ge 0$ and $\varphi \in [0, 2\pi)$. Moreover, it can be shown [6] that when $\alpha < 1$, $A_{\alpha,\varphi}(\omega_x,\omega_y)$ is positive on $[-\pi,\pi)^2 \setminus (0,0)$ and thus $\mathcal{S}(\omega_x,\omega_y)$ has no singularity except at (0,0).

We now study the behaviour of $S(\omega_x, \omega_y)$ in the neighborhood of (0, 0) in a polar coordinate system

$$\begin{cases} \omega_x = \omega_r \cos \omega_\theta \\ \omega_y = \omega_r \sin \omega_\theta \end{cases}$$

where $\omega_r \geq 0$. As $\omega_r \to 0$, we obtain

$$S(\omega_r \cos \omega_\theta, \omega_r \sin \omega_\theta) \sim \frac{2^{2d} \sigma_w^2}{\omega_r^{4d} \left[1 + \alpha - 2\alpha \cos^2\left(\omega_\theta - \frac{\varphi}{2}\right)\right]^2}.$$
 (5)

The process defined by (3) has long-dependence properties only if its power spectrum density is divergent at (0,0), which requires d > 0. On the other hand, in order to have a well-defined process (in the mean square sense), one must impose

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \mathcal{S}(\omega_x, \omega_y) d\,\omega_x \, d\omega_y < \infty$$

which, as $\alpha < 1$, implies 4d - 1 < 1 and therefore d < 1/2. Finally, the domain of validity for d is the interval (0, 1/2).

The interpretation of the model parameters is illustrated in Fig. 2.

3. IDENTIFICATION METHOD

We have already studied the case where the driving noise w(n,m) is Gaussian [5]. In this work, we are interested in a non-Gaussian field generated by a Gaussian mixture¹:

$$w(n,m) \sim (1-\varepsilon) \mathcal{N}(0,\sigma_0^2) + \varepsilon \mathcal{N}(0,\sigma_1^2),$$

with $\varepsilon \in [0, 1]$.

A method based on a direct maximization of the likelihood function for estimating the parameters of the model is complicated as it leads to the optimization of a function of 6 variables. The use of the Estimation- Maximization (EM) algorithm was found to be successful in the Gaussian case ([5, 6]). However, this algorithm becomes intractable for a Gaussian mixture because of the longdependence properties of the considered model. For the identification of this model, we propose a statistical method consisting in two steps: 1) We estimate the fractional parameter d, the parameters characterizing the anisotropy (α) and the directionality (φ) of the field and the global variance of the generating noise, $\sigma_w^2 = (1 - \varepsilon) \sigma_0^2 + \varepsilon \sigma_1^2$, by minimizing the mean-square error between the log-periodogram of the image and its log-spectrum density. The corresponding function to be minimized is given by:

$$E(\sigma_w^2, d, \alpha, \varphi) = \sum_{(k,l)\neq(0,0)} \left\{ \log P_N(\omega_{k,N}, \omega_{l,N}) + C_E - \log \sigma_w^2 + 2(d-1) \log I(\omega_{k,N}, \omega_{l,N}) + 2 \log A_{\alpha,\varphi}(\omega_{k,N}, \omega_{l,N}) \right\}^2$$
(6)

where P_N denotes the periodogram of the $N \times N$ image, C_E is the Euler's constant and $\omega_{k,N} = (2\pi k)/N$, $k \in \{0, \ldots, N-1\}$. As this criterion linearly depends on d and $\log \sigma_w^2$ and nonlinearly on α and φ , the estimation problem reduces to the iterative optimization of a function of 2 variables.

2) Based on these parameter estimates, we can realize an estimation of the parameters of the driving noise w(n,m). Two methods have been envisaged to perform this task.

• *Forward modeling approach.* The cumulants of the observed image u(n, m) are matched to their theoretical expressions. For example, we can use the relations

$$\operatorname{cum}_{4} u = 3 \varepsilon (1 - \varepsilon) (\sigma_{1}^{2} - \sigma_{0}^{2})^{2} \sum_{(k,l) \in \mathbb{Z}^{2}} h(k,l)^{4}$$
$$\operatorname{cum}_{6} u = 15 \varepsilon (1 - \varepsilon) (1 - 2\varepsilon)$$
$$\times (\sigma_{0}^{2} - \sigma_{1}^{2})^{3} \sum_{(k,l) \in \mathbb{Z}^{2}} h(k,l)^{6}$$

where h(k, l) denotes the impulse response of the synthesis filter. This impulse response can be deduced from the estimations performed in Step 1. A third equation is provided by the expression of the variance. The variance of the driving noise can be obtained from the previously estimated value of $\log \sigma_w^2$

¹The normal distribution with mean μ and standard deviation σ is denoted by $\mathcal{N}(\mu, \sigma^2)$.

or it can be re-estimated using the autocorrelation sequence $\gamma(k, l)$ of u(n, m) when $(k, l) \neq (0, 0)$. In this latter case, one could expect the method to be robust with respect to a Gaussian noise added to the observed image.

• *Inverse modeling approach.* We can invert the synthesis filter in order to obtain an estimation of the driving noise. This inversion is possible due to the boundedness of the frequency response of the inverse filter:

$$0 < D^d_{\alpha,\omega}\left(e^{j\omega_x}, e^{j\omega_y}\right) \le (1+\alpha)2^d.$$

We then estimate the mixture parameter ε and the variances σ_0^2 and σ_1^2 by using a method of moments. This is based on the calculation of the *p*-th order asbolute mean values of w(n,m):

$$\mathbb{E}\{|w(n,m)|^p\} = C_p[(1-\varepsilon)\sigma_0^p + \varepsilon\sigma_1^p]$$

where C_p is a constant depending on p. Closed form expressions of the estimated parameters are obtained from the first, second and third absolute mean values of the estimated driving noise.

4. EXPERIMENTAL RESULTS

In Fig. 1 we present a realization of a non-Gaussian anisotropic field and in Table 1 the estimation of the fractional model parameters resulting from a Monte Carlo study carried out over 100 realizations of the process (256×256 pixels). As the proposed estimation methods do not explicitly take into account a possible measurement/modeling noise in the observed image, the robustness of the estimators with respect to a $\mathcal{N}(0, \sigma_b^2)$ additive noise has also been evaluated in this table.

We have also compared the performances of the two proposed estimation methods for the parameters of the driving noise in Table 2. The inverse

	Par.	d	α	φ
	Value	0.3	0.7	$\pi/4$
$\sigma_b = 0$	mean	0.3002	0.6992	0.7830
	std	0.0042	0.0060	0.0094
$\sigma_b = 0.7$	mean	0.2995	0.6975	0.7837
	std	0.0042	0.0060	0.0094

Table 1: Performance of the proposed estimation method for the parameters of the anisotropic fractional noise model.

modeling approach is denoted by "IM", while the foward modeling one is designated by "FM". We remark the good performances of the

	Par.	ε	σ_0	σ_1
	Value	0.4	1	15
IM	mean	0.3997	1.0050	15.0020
$\sigma_b=0$	std	0.0042	0.0292	0.0999
	mean	0.3998	1.2355	15.0266
$\sigma_b = 0.7$	std	0.0042	0.0302	0.1015
FM	mean	0.4035	1.0217	14.9759
$\sigma_b=0$	std	0.0309	1.1530	0.3653
	mean	0.4037	1.1488	15.0193
$\sigma_b = 0.7$	std	0.0306	1.1791	0.3618

Table 2: Comparison of the performances of the forward and inverse modeling estimation methods for the Gaussian mixture parameters.

inverse modeling approach in the presence of additive noise. This may be explained by the boundedness of the frequency response of the inverse synthesis filter and the low orders of moments used to estimate the parameters. The forward modeling method leads to an unbiased estimation of the parameters, but it exhibits a higher mean square estimation error.

Note that our method may be extended to semiparametric estimation where the input distribution is arbitrary. Indeed, one can approximate many usual distributions by a mixture of Gaussians.



Figure 1: A realization of the proposed non-Gaussian anisotropic model ($d = 0.3, \alpha = 0.7, \varphi = \pi/4$).

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Figure 2: Power spectrum density of the anisotropic noise for different values of the parameters: $\varphi = -\pi/5, \alpha = 0.97, d = 0.3$ (up) and $\varphi = \pi/5, \alpha = 0.57, d = 0.3$ (down). The angle $\varphi/2$ is measured in inverse trigonometric sense with respect to the vertical axis.