HAUSDORFF DISTANCE AND FRACTAL DIMENSION ESTIMATION BY MATHEMATICAL MORPHOLOGY REVISITED

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ABSTRACT

This work[1] demonstrates that the distance measuring the likelihood of the graphs of two functions, usually referred as Hausdorff distance between functions and widely used in function approximation tasks and signal processing, can be calculated efficiently using grey - scale morphological operations even in the case of noncontinuous (discrete as well as nondiscrete) functions. Also we have presented a generalization of the Bouligand definition of fractal dimension working also in the case of noncontinuous objects. Our results are based on the usage of the notion of a closed graph a function, as defined by Sendov.

1. INTRODUCTION

Image and signal measurements are of particular interest for featuring regions for both segmentation and classification purposes. This paper works with signal and image features which can be evaluated by morphological operations. One of the most useful features in signal and image processing is the fractal dimension - especially in the case of medical signal processing or processing of texture images in aerial photography, X-ray medical tomography and quality inspection of surface maintenance [7] .

As mentioned in [9] , in many approximation problems it is useful to work with the so called Hausdorff distance between functions. We show that the distance measuring the likelihood of the graphs of two functions, usually referred as Hausdorff distance between functions and widely used in function approximation tasks and signal processing, can be calculated efficiently using grey-scale morphological operations. The main profit is that we approximate the original function – signal or image – by a simpler one in the class of piecewise polygons, Bezier functions, cubic splines etc., which graph shape is close to the original graph shape. Also, our results are shown to be useful in signal sampling via morphological strategy [4] . To obtain a complete description of a signal or an image it must be analyzed over a complete range of spatial scales, which is expressed by the term multiresolution representation. A multiresolution representation is a sequence of images in which each image is a filtered and subsampled copy of its predecessor. A natural criterium for stopping the sampling procedure is when the Hausdorff distance between the real image and its sampled version becomes small enough, or alternatively, when the Hausdorff distance between a sampled version of the image and its predecessor becomes small enough. Following Sendov [9], in our work we define the closed graph of a signal or an image as an interval- valued function represented by the upper and the lower Baire function of the original signal. Further on we shall refer to the objects under consideration as signals, although our results are applicable to spaces with dimension higher than one, for instance to two and three dimensional visual scenes.

Many natural texture images present structures at all scales and show high degree of roughness. A key parameter for such images, or signals, is their fractal dimension – a real positive number (usually noninteger) which is widely used as texture descriptor in image processing [7] . We have presented a generalization of the Bouligand definition of fractal dimension working also in the case for noncontinuous discrete and nondiscrete signals and images. This definition is based also on the usage of the notion of a closed graph a function.

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2. MORPHOLOGICAL OPERATIONS – PRELIMINARIES

Let us assume that our objects lie in a linear space M. Considering two subsets of M, namely A and B the following operations are used:

- Minkowski addition of A and B, $A \oplus B$ is defined as $A \oplus B = \bigcup_{b \in B} A + b = \{a + b \,|\, a \in A, b \in B\}$, and Minkowski difference of A by B, $A \ominus B$ is defined as $A \ominus B = \bigcap_{b \in B} A - b = \{x \in M | B + x \subseteq A\}$. For the properties of these operations see [3] .
- Opening of A by B is defined as $A \circ B = (A \ominus B) \oplus B$, while closing of A by $B$is defined as $A \bullet B = (A \oplus B) \ominus B$.

Let us now consider as M the m-dimensional Euclidean space R^m. Here and henceforth $B_r^m(x)$ denotes the closed ball in the m-dimensional space (disk in two dimensional space) with centre x and radius r. Sometimes we miss the upper index when there is no confusion with dimensions. We shall omit the centre of the ball when the ball is centered at the origin. Then if X is a compact set from R^m, then we shall call the compact set $X \oplus B_r$ parallel body of X with thickness r. If X is a curve in the plane, then the parallel body will be referred to as a parallel ribbon.

If P and Q are non-empty compact sets in R^n. Then the Hausdorff distance between P and Q [10] is defined as

$$\mathrm{dist}\,(P, Q) = \max\{h_1(P, Q)\,, h_2(P, Q)\},$$

where $h_1(P, Q) = \max\{d(x, P) \,|\, x \in Q\}$, and $h_2(P, Q) = \max\{d(x, Q) \,|\, x \in P\}$. By $d(x, A)$ we denote the Euclidean distance between the point x and the set A, namely $d(x, A) = \inf\{|x - a| \,|\, a \in A\}$, and $|z|$ is the Euclidean norm of the vector $z \in \mathrm{R}^m$.

It is well known (see for instance [10] or [1]) that

$$\mathrm{dist}\,(P, Q) = \inf\{\varepsilon \,|\, Q \subseteq P \oplus B_\varepsilon(0)\,, P \subseteq Q \oplus B_\varepsilon(0)\}. \tag{1}$$

Let consider the grey - scale morphological operations in $\mathrm{Fun}(M)$ – the family of all functions mapping the linear space M to the compactified real line $\bar{\mathrm{R}} = \mathrm{R} \cup \{\infty, -\infty\}$. The basic morphological operations – erosions and dilations – are defined by

$$(\delta_g(f))(x) = \sup_{x \in M} (f(x - h) + g(h)), \tag{2}$$

$$(\varepsilon_g(f))(x) = \inf_{x \in M} (f(x + h) - g(h)), \tag{3}$$

supposing that $s + t = -\infty$ if $s = -\infty$ or $t = -\infty$, and $s - t = \infty$ if $s = \infty$ or $t = -\infty$. Then every translational - invariant dilation is given by (2), and every translational - invariant erosion – by (3). Further we shall denote $\delta_g(f)$ by $f \oplus g$, and $\varepsilon_g(f)$ by $f \ominus g$. These operations are named grey - scale morphological operations [10] and are widely used in the analysis of grey - scale images.

3. COMPUTING THE HAUSDORFF DISTANCE BETWEEN FUNCTIONS BY GREY- SCALE MORPHOLOGICAL OPERATIONS

As mentioned in [9], in many approximation problems it is useful to work with the so called Hausdorff distance between functions. The main profit is that we approximate the original function by a simpler one (piecewise polygons, Bezier functions, splines etc.) which graph shape is close to the original graph shape. We shall consider only functions with bounded domain. But before setting the main result we need some additional notions and notations.

In [3] it is shown that there exists a simple relation between binary and grey - scale operations. So, let define the set named umbra $U \subseteq M \times \bar{\mathbf{R}}$ with the property: the point $(x, t) \in U$ if and only if for every $s < t$ it follows that $(x, s) \in U$. Given a function $f(x)$ from $\mathrm{Fun}(M)$, we can define an umbra

$$\Gamma(f) = \{(x, t) \in M \times \bar{\mathbf{R}} | \, t \le f(x)\}.$$

Conversely, given an umbra U, we can construct the function

$$f_U(x) = \sup\{t \in \mathbf{R} | (x, t) \in U\},$$

such that $\Gamma(f_U)$ is namely the umbra set U. We can define also an upper closure of the set $A \subseteq M \times \bar{\mathbf{R}}$ as the minimal umbra set containing A, which will be denoted by $\Upsilon(A)$, i.e. $\Upsilon(A)$ is the intersection of all umbrae containing A. It is evident that the $\Upsilon()$ operation is a closing. In [3] also the class of pre-umbra sets is considered: The set H is a pre-umbra if for every point $(x, t) \in H$ and for every $s < t$ the inclusion $(x, s) \in H$ holds. It is clear that the Minkowski sum of two umbrae is a pre-umbra. Also, if H is a pre-umbra, then $\mathrm{cl}\,(\Upsilon(H)) = \mathrm{cl}\,(H)$. Here and henceforth, we denote by $cl(X)$ the topological closure of the set X. The last equality follows from the fact that H consists of closed and non-closed rays and therefore $\Upsilon(H)$ consists of closed rays. Then the equality follows immediately from the fact that the sequence of real numbers

$\{x_n\}$ converges just when $\{x_n - \frac{1}{n}\}$ converges and vice versa and they have a common limit. Also, if A is a topological space and A is closed in $M \times \bar{\mathbf{R}}$, then $\Upsilon(A)$ is closed in $M \times \bar{\mathbf{R}}$ as well.

In [3] the following theorem is proved:

Theorem 1. $, (f \oplus g) = \Upsilon(, (f) \oplus , (g))$, $, (f \ominus g) = , (f) \ominus , (g)$.

Let now $M = \mathbf{R}^n$ and let us consider the closed graph of the function $f : \mathbf{R}^n \mapsto \bar{\mathbf{R}}$ denoted by $\Phi(f)$ (see [9]) – it is defined as an interval valued function $(\Phi(f))(x) = [(I(f))(x), (S(f))(x)]$, where $(I(f))(x)$ and $(S(f))(x)$ are the lower and upper Baire functions of f respectively [9]:

$$(I(f))(x) = \lim_{\delta \to +0} \inf\{f(t) | t \in B_\delta(x) \cap \mathrm{dom}\,(f)\}$$
$$(S(f))(x) = \lim_{\delta \to +0} \sup\{f(t) | t \in B_\delta(x) \cap \mathrm{dom}\,(f)\}.$$

Or equivalently,

$$(I(f))(x) = \liminf_{t \to x} f(t)$$
$$(S(f))(x) = \limsup_{t \to x} f(t).$$

Here by $\mathrm{dom}\,(f)$ we denote the domain of f –

$$\mathrm{dom}\,(f) = \{x \mid \infty > f(x) > -\infty\}.$$

The closed graph can be considered also as a closed set in $\mathrm{R}^n \times \bar{\mathrm{R}}$.

Lemma 2. If $f : \mathbf{R}^n \mapsto \bar{\mathbf{R}}$, then $\mathrm{cl}\,(, (f)) = \Upsilon(\Phi(f))$.

Proof. Let $y \in \Upsilon(\Phi(f))$. Without lack of generality we may assume that $y = (S(f))(x)$ for any x. We shall show that $(x, y) \in \mathrm{cl}\,(, (f))$.

Consider a sequence of closed balls $B_{\delta_i}(x)$, $\lim_{i\to\infty} \delta_i = 0$. Let us define the numbers $S_i = \sup\{f(t) | t \in B_{\delta_i}(x) \cap \mathrm{dom}\,(f)\}$. Then there exist points $x_i \in B_{\delta_i}(x) \cap \mathrm{dom}\,(f)$ such that $S_i \geq f(x_i) \geq S_i - \frac{1}{n}$. Therefore $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} f(x_i) = (S(f))(x) = y$. Since the points $(x_i, f(x_i))$ are from $, (f)$, then $(x, y) \in \mathrm{cl}\,(, (f))$ and $\Upsilon(\Phi(f)) \subseteq \mathrm{cl}\,(, (f))$.

Conversely, let us now choose a point $(x, y) \in , (f)$ and suppose for simplicity that $y = f(x)$. Then $(I(f))(x) \leq f(x) \leq (S(f))(x)$ which means that $, (f) \subseteq \Upsilon(\Phi(f))$. From the upper semicontinuity of the upper Baire function (see [9]) it follows that $\Upsilon(\Phi(f))$ is closed. Therefore $\mathrm{cl}\,(\Upsilon(, (f))) \subseteq \Upsilon(\Phi(f))$, which proves the lemma. □

Corollary 3. $, (S(f)) = \mathrm{cl}\,(, (f)) = \Upsilon(\Phi(f))$.

In [9] Sendov defines the Hausdorff distance between the functions f and g as $\mathrm{d}(f, d) = \mathrm{dist}\,(\Phi(f), \Phi(g))$, where dist is the ordinary Hausdorff distance between sets.

In this work we demonstrate that the distance d can be easily expressed by grey - scale morphological operations, which enables its simpler computation using only the values of the upper Baire function.

Sendov's definition of Hausdorff distance between functions is more general and more precise than that studied in [1] and correctly treats the function discontinuities. For discrete signals both definitions lead to the same result.

Let $x \in \mathbf{R}^n$. For every $\varepsilon > 0$ we define the function $b_\varepsilon(x) = \max\{y | (x, y) \in B_\varepsilon^{n+1}(x)\}$ with domain the ball $B_\varepsilon^n(0)$.

Theorem 4. $\mathrm{d}(f, g) = \sup\{\varepsilon | S(f) \leq S(g) \oplus b_\varepsilon, S(g) \leq S(f) \oplus b_\varepsilon\}$.

Proof. From Theorem 1 it follows that $, (S(f)) \oplus b_\varepsilon) = \Upsilon(, (S(f)) \oplus , (b_\varepsilon))$ From simple geometrical reasons it follows also that $, (S(f)) \oplus , (b_\varepsilon) = , (S(f)) \oplus B_\varepsilon^{n+1}(0)$. Since $, (S(f))$ is closed (see Lemma 2), and $B_\varepsilon^{n+1}(0)$ is compact, then their Minkowski sum is topologically closed. Also, the sum of two umbrae is a pre-umbra, but when a set is a topologically closed pre-umbra, it is an umbra. Therefore $, (S(f) \oplus b_\varepsilon) = , (S(f)) \oplus B_\varepsilon(0)$. For the two functions u and v we have $u \leq v$ just when $, (u) \subseteq , (v)$. Then from the Corollary of Lemma 2, $, (S(f)) = \Upsilon(\Phi(f))$. Therefore the inequality $S(g) \leq S(f) \oplus b_\varepsilon$ is equivalent to the inclusion $\Upsilon(\Phi(g)) \subseteq \Upsilon(\Phi(f)) \oplus B_\varepsilon(0)$ which is satisfied evidently when $\Phi(g) \subseteq \Upsilon(\Phi(f)) \oplus B_\varepsilon(0)$. Then the proof of our theorem follows immediately from the definition of Hausdorff distance between sets. □

4. ESTIMATION OF THE FRACTAL DIMENSION OF SIGNALS

The dimension of a real object refers to properties known as length, area and volume. An object having length is said to be one - dimensional; if it has area but its volume is zero it is of two dimensions; and it has only a volume - it is of three dimensions. Or speaking more precisely, a geometrical object is n-dimensional, if n is the least least real number parameter used to determine continuously the points of the configuration of the object [12]. Definitions of dimensions can be given also to compact objects having infinite area, area or volume. These sets are known to be fractals. Or in other words, fractals are sets whose geometry follows the Hausdorff concept of dimension [2].

Usually, a set is called to be a fractal, if its Hausdorff - Besicovitch dimension is strictly greater than its topological dimension [2, 12]. However, the Hausdorff - Besicovitch dimension is usually difficult to estimate in practice. Another dimension characteristics, much more easier to compute, is the so called Minkowski - Bouligand dimension.

First Minkowski suggested to measure the length of two-dimensional curve by measuring the area $S\delta$ of its parallel ribbon by a disk B_δ of radius δ, dividing this area by the thickness 2δ and calculating the limit of this quotient when δ tends to zero. For curves with finite length it is easy to demonstrate that the limit $\lim_{\delta\to 0}\left[\frac{\log S_\delta}{\log \delta}\right]$ is equal to one.

Therefore for arbitrary compact set X in R^n Bouligand defined the following limit:

$$D(X) = \lim_{\delta\to 0}\left(D_T(X) - \frac{\log\ \mu^{n+1}(X\oplus B_\delta^{n+1})}{\log\ \delta}\right), \quad (4)$$

where $D_T(X)$ means the topological dimension of X, and μ^k means the k-dimensional volume (Lebesgue measure).

D is known now as Minkowski - Bouligand dimension of the set X, and for fractal sets (i.e. sets with self similarity or statistical self-similarity) $D_T(X) < D(X) \leq D_T(X) + 1$, while for nonfractal sets $D = D_T$.

Let us now consider the interesting from signal processing point of view case, when X is the graph of a bounded function $f : P \mapsto \mathrm{R}$, and P is compact connected subset of R^n, and its boundary ∂P is measurable, i.e. $\mu^{n-1}(\partial P) < \infty$. Then from the sigma - additivity of the Lebesgue measure, it follows easily that in (4) we may substitute the volume $X \oplus B_\delta^{n+1}$ by the volume of the truncated parallel body of X, defined as $(X\oplus B_\delta^{n+1}) \cap (P \times (-\infty, \infty))$. If f is continuous, then the truncated parallel body of its graph is a compact set in R^{n+1}.

In Theorem 1 in [5] it is proved that if $f : P \mapsto \mathrm{R}$ is a continuous function defined on such a set P as described above, and if b_ε is the function defined in the previous section, then the $n+1$-dimensional volume of the truncated parallel body of the graph of f with thickness ε can be calculated through Lebesgue integration as $\int_P (f\oplus b_\varepsilon - f\ominus b_\varepsilon)$. The proof is made for one - dimensional signals, but it can be repeated without any problems and the result holds true for any Euclidean space R^n – see for instance [12] for the two - dimensional case.

Let consider our image, or signal, as a function f and let for $\varepsilon > 0$ define the function $f^\varepsilon = S(f)\oplus b_\varepsilon - I(f)\ominus b_\varepsilon$. Then denoting for simplicity $P = \mathrm{dom}(f)$ we can define by Lebesgue integration the following volume - like quantity $V_\varepsilon(f) = \int_P f^\varepsilon$. Then, for both continuous and noncontinuous signals and images we define the number $D(f) = \lim_{\varepsilon\to 0}\left(D_T - \frac{\log\ V_\varepsilon(f)}{\log\ \varepsilon}\right)$, where D_T means the topological dimension of the closed graph of the our image f. For instance, D_T equals one in the case of signals, and two in the case of visual scenes. This is a generalization of the Bouligand definition of fractal dimension working also in the case for noncontinuous objects. If our signal f is statistically self-similar there is no need to compute the limit. Following directly the proof of Theorem 1 in [5] it follows that $D(f)$ is namely the Minkowski - Bouligand dimension of the closed graph $\Phi(f)$ of the function f when the domain of f is compact, connected set with measurable border. The new proof slightly differs from the proof of Maragos and Sun, and exploits the upper semicontinuity of $S(f)$ and the lower semicontinuity of $I(f)$ instead of the continuity of f in the original result of Maragos and Sun in [5].

5. CONCLUSION AND FUTURE RESEARCH

We have shown that the notion of a closed graph of a continuous -time signal using its upper and lower Baire function is useful to calculate features like Hausdorff distance and fractal dimension.

Further we shall attempt to derive convergence criteria between the fractal dimension of a signal and the discrete analogs of fractal dimension (as used for instance in [7]) of its morphological samples calculated discretely.

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