2-DIMENSIONAL STOCHASTIC RESONANCE IN A DISCRETE TIME NONLINEAR SYSTEM WITH A NOISY ELLIPTIC INPUT

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ABSTRACT

This paper deals with stochastic resonance. This nonlinear physical phenomenon generally occurs in bistable systems excited by a random noise plus a sine. Such systems force cooperation between the input noise and the input sine: Provided a fine tuning between the power noise and the dynamics, the system reacts periodically. The interesting fact is that the local output signal-to-noise ratio presents a maximum when plotted against the input power noise. In this paper we recall the results for the discrete-time one-dimensional nonlinear AR(1) systems. We then extend the study to a particular 2-dimensional nonlinear system.

1. INTRODUCTION

Stochastic Resonance (SR) is a physical phenomenon occuring generally in bistable dynamical systems excited by random noise and a sine. To fix the idea, assume that a particle is moving in a bistable potential. Under the assumption of strong friction, the particle will fall and stay in one well of the potential. If the particle is then excited by a random noise, the particle will have a non-zero probability to hop from one well to the other. Now, if we add a sinusoid in the input, the potential will be modulated, and provided a fine tune of the parameters (sine and noise amplitude), a cooperative effect between the sinusoid and the noise takes place: The particle will jump from one well to the other at the frequency of the sinusoid. The interesting fact is that the output Signal-to-Noise Ratio (SNR) at the frequency of the sinusoid presents a maximum when plotted against the variance of the input noise (see Figures 1 and 2). This fact has led several researchers to examine the ability of SR to detect [1, 12] or amplify small amplitude periodical signal [8].

The theory of SR in continuous time systems is difficult. However, under some assumptions, several approximate theories exist that explain that phenomenon [9]. We have examined in [13] SR in a discrete time context, through the nonlinear AR(1) model:

$$\left\{ \begin{array}{lcl} x_n & = & \Phi(x_{n-1}) + b_n + \varepsilon_n \\ y_n & = & c\,\mathrm{sign}(x_n) \end{array} \right. \qquad (1)$$

where b_n is an independent identically distributed (iid) noise, ε_n is a weak sine of amplitude ε and of frequency λ_0 and where $\Phi(x)$ is taken bistable, where $\pm c$ are the two stable equilibrium points. In this paper we will recall the essential theoretical results. Then we will present some further results on the simple case $\Phi = c\,\mathrm{sign}$. Finally we will extend the results on the 2-dimensional system of the form (1), where x_n will be a 2-dimensional vector and where for $x = [x_0\;\; x_1]^t$, Φ will be defined as

$$\Phi(x) = c[\mathrm{sign}(x_0)h(x_1)\;\; \mathrm{sign}(x_1)h(x_0)]^t$$

where $h(x) = 1$ if $x \in [-\alpha c\;\; \alpha c]$ and 0 elsewhere, $\alpha > 1$. The two components of b_n will be assumed independent and iid, and the two components of ε_n will be two weak sinusoids of the same frequency λ_0, of amplitude respectively ε_0 and ε_1, and of phase respectively φ_0 and φ_1. Considering these sines as representative of a 2-dimensional motion, ε_n will then represent an elliptic motion.

2. BRIEF OVERVIEW OF THE METHOD FOR THE ONE-DIMENSIONAL CASE.

In the one-dimensional system, the iid property of b_n implies that x_n and then y_n are Markovian. Thus the Chapmann-Kolmogorov equation can be written to determine the recursion on the probability density function (pdf) of x_n, $f_x(x,n) = \int_{\mathbb{R}} f_b(x - \Phi(y) - \varepsilon_n)\, f_x(y, n-1)\, \mathrm{d}y$, where f_b and $f_x(.,n)$ are the pdf of b_n and x_n. We then make a Taylor expansion (order 1) of $f_b(x - \Phi(y) - \varepsilon_n)$ in $x - \Phi(y)$ and we show in [13] that the pdf of x_n is asymptotically composed of the pdf obtained in the absence of the sine ($\varepsilon = 0$) plus a modulation term. Then the probability vector of y_n and the transition matrix from y_{n-1} to y_n can easily be evaluated. Due to the Markov property the transition matrix from y_n to

y_{n+k} is simply the product of the one-step transition matrices. Then using the transition matrix from y_n to y_{n+k} and the probability vector of y_n, the correlation function of signal y_n, $\Gamma(n,k) = E[y_{n+k}y_n]$, can be calculated. It leads to the zero-cyclic correlation function $\Gamma(k) = \langle \Gamma(n,k)\rangle_n$ (where $\langle . \rangle_n$ represent an average in n): $\Gamma(k) \approx c^2\beta^k + \frac{\varepsilon^2|\chi(\lambda_0)|^2}{2}\cos(2\pi k\lambda_0)$. Parameter β and the susceptibility $\chi(\lambda)$ depends on the system and on the input noise (see [13]). Hence the local output SNR at the frequency of the sine λ_0, defined as the ratio between the power of the output sine and the power spectral density function of the output noise at λ_0, is given by $SNR = \frac{\varepsilon^2|\chi(\lambda_0)|^2(1+\beta^2-2\beta\cos(2\pi\lambda_0))}{4c^2(1-\beta^2)}$.

For the simple system $\Phi = c\,\text{sign}$, the parameters are simply $\beta = F_b(c) - F_b(-c)$ and $\chi(\lambda) = \frac{2cf_b(c)}{1-\beta\exp(-2\imath\pi\lambda)}$ (f_b and F_b are respectively the probability and the cumulative density function of b_n). Then we can evaluate the local output SNR at frequency λ_0. This SNR is plotted against the input noise power σ^2 in figure 1 when the input noise is Gaussian and in figure 2 when the input noise is uniform. This figures exhibit the SR phenomenon.

3. EXTENSION TO A PARTICULAR 2-DIMENSIONAL CASE

Multidimensional systems showing SR have already been studied [3, 11], but the output of the studied systems are one-dimensional.

We have not enough place here to present the development of the calculus in details, but in the general d-dimensional case, the method is the same than that use for the one-dimensional case [13]. We just give the fundamental elements of the calculus for the particular 2-dimensional case (1) where x_n is a 2-dimensional vector and where for $x = [x_0 \;\; x_1]^t$, Φ is defined as

$$\Phi(x) = c[\text{sign}(x_0)h(x_1) \;\; \text{sign}(x_1)h(x_0)]^t \tag{2}$$

where $h(x) = 1$ if $x \in [-\alpha c \;\; \alpha c]$ and 0 elsewhere, $\alpha > 1$. The two components of b_n are assumed independent and iid, and the two components of ε_n are two weak sinusoids of the same frequency λ_0, of amplitude respectively ε_0 and ε_1, and of phase respectively φ_0 and φ_1. The pdf of b_n is assumed centro-symmetric.

In this 2-dimensional case, the pdf of x_n is recursively written using that of x_{n-1} (Chapmann-Kolmogorov or marginal density)

$$f_x(x,n) = \int_{\mathbb{R}^2} f_b(x - \Phi(y) - \varepsilon_n)\, f_x(y,n)\, \mathrm{d}y \tag{3}$$

As done in the one-dimensional case in [13], $f_b(x - \Phi(y) - \varepsilon_n)$ is expanded in $x - \Phi(y)$, i.e. $f_b(x - \Phi(y) - \varepsilon_n) \approx f_b(x - \Phi(y)) - \varepsilon_n^t f_b'(x - \Phi(y))$ (where $f_b'(u)$ is the vector containing the partial derivates of f_b).

As in the one-dimensional case [13], it can then be shown that $f_x(.,n)$ is asymptotically of the form

$$f_x(.,n) = f_{\text{wm}} + \varepsilon_n^t f_\varepsilon + \mu_n^t f_\mu \tag{4}$$

where f_{wm} is the pdf of x_n when there are no sine in the input and where μ_n is the quadrature of ε_n. The functions $f_\varepsilon(x)$ and $f_\mu(x)$ are 2-dimensional and $\varepsilon_n^t f_\varepsilon + \mu_n^t f_\mu$ is then the contribution of the sine to the pdf. f_{wm} is given by the the eigenequation

$$f_{\text{wm}}(x) = \int_{\mathbb{R}^2} f_b(x - \Phi(y))\, f_{\text{wm}}(y)\, \mathrm{d}y \tag{5}$$

and the other terms are given by $f_\varepsilon + i f_\mu = f_{\text{mod}}$,

$$\begin{aligned} f_{\text{mod}}(x) \;&=\; -\int_{\mathbb{R}^2} f_b'(x - \Phi(y))\, f_{\text{wm}}(y)\, \mathrm{d}y \\ &+ \mathrm{e}^{-2i\pi n\lambda_0}\int_{\mathbb{R}^2} f_b(x - \Phi(y))\, f_{\text{mod}}(y)\, \mathrm{d}y \end{aligned} \tag{6}$$

Using the pdf of x_n the probability vector of y_n,

$$p_y(n) = \begin{bmatrix} \Pr[\, y_n^t = [-c \;\; -c]\,] \\ \Pr[\, y_n^t = [+c \;\; -c]\,] \\ \Pr[\, y_n^t = [-c \;\; +c]\,] \\ \Pr[\, y_n^t = [+c \;\; +c]\,] \end{bmatrix}$$

can be determined. This vector is here of the form $p_y(n) = \theta_y + m_y(n)$ where θ_y is the probability vector in the absence of the modulation, and where $m_y(n)$ represent the (small) contribution of the modulation.

As done in [13] in the one-dimensional case, the transition matrix $P(n+1,n)$ from y_n to y_{n+1} can be evaluated. This matrix is of the form $P_y(n+1,n) = R_y + M_y(n+1,n)$ where R is the transition matrix in the absence of the modulation, and where $M(n+1,n)$ represent the (weak) contribution of the modulation.

b_n is iid, then y_n is Markovian: The transition matrix $P_y(n+k,n)$ from y_n to y_{n+k} is simply the product of the one step transition matrices.

Finally, using the probability vector $p_y(n)$ and the transition matrix $P_y(n+k,n)$, we can easily determined the zero-cyclic correlation function of the output, defined as $\Gamma_y(k) = \langle E[y_{n+k}\, y_n^t]\rangle_n$. This correlation function is of the form

$$\begin{aligned} \Gamma(k) \;\;&\approx\;\; c^2 \begin{bmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{bmatrix}^k \\ &+ \begin{bmatrix} \frac{\varepsilon_0^2|\chi_0(\lambda_0)|^2}{2} & 0 \\ 0 & \frac{\varepsilon_1^2|\chi_1(\lambda_0)|^2}{2} \end{bmatrix} \cos(2\pi k\lambda_0) \end{aligned}$$

$$+ \frac{\varepsilon_0\varepsilon_1 |\chi_0(\lambda_0)\chi_1(\lambda_0)|}{2} \times \begin{bmatrix} 0 & \cos(2\pi k\lambda_0 + \varphi(\lambda_0)) \\ \cos(2\pi k\lambda_0 - \varphi(\lambda_0)) & 0 \end{bmatrix}$$

where parameters β_0 and β_1, susceptibilities $\chi_0(\lambda)$ and $\chi_1(\lambda)$ depend of the input noise b_n (both the two components, $\varphi(\lambda_0) = \varphi_0 - \varphi_1 + \text{Arg}(\chi_0(\lambda_0)) - \text{Arg}(\chi_1(\lambda_0))$).

Two local SNR (SNR_0 and SNR_1) at frequency λ_0 can be defined as for the one-dimensional case, using the diagonal terms.

In the particular case studied here, Φ is constant in nine domains of $\mathbb{R}^2$. Then the eigenequation giving f_{wm} reduces to a matricial eigenequation as for the SETAR case (see [13]). The equation giving f_{mod} also reduces to a matricial equation.

We use then these matricial equations to reduce a little the complexity of the calculus. The complete calculus are quit long and the formula giving parameters β_0, β_1, χ_0 and χ_1 are too complex. There is no interest to presented them here.

The numerical study has been done (using exactly the same scheme than that used to study the SETAR models in [13]) for varying variances of the first and second component of b_n, σ_0^2 and σ_1^2. Figures 3 and 6 then depicts $|\chi_0(\lambda_0)|$ as a function of σ_0 and σ_1 in the Gaussian and Uniform cases. This shows the amplification of the first component of the input sine, and exhibit the SR phenomenon, as well in σ_0 as in σ_1. This result is confirmed in figures 4, 5, 7 and 8. Hence such a system can create 2-dimensional SR. Notice that the form of $|\chi_1(\lambda_0)|$ here is simply the symmetric of $|\chi_0(\lambda_0)|$ against $\sigma_0 = \sigma_1$. The SNR are not depicted here, but show the same properties as the susceptibilities.

4. CONCLUSION.

We have recalled in this paper that SR, that is initially a physical phenomenon, can possibly be used in signal processing to amplify small noisy sines. We than suggest that this kind of discrete-time SR can be extended to d-dimensional SR, studying a particular 2-dimensional AR(1) system. We then show that such an extension gives some interesting results. Indeed, it can particularly shown that the first component of the elliptic input can be amplified by the second component of the input noise! We currently, study the extension to general d-dimensional AR(1) systems.

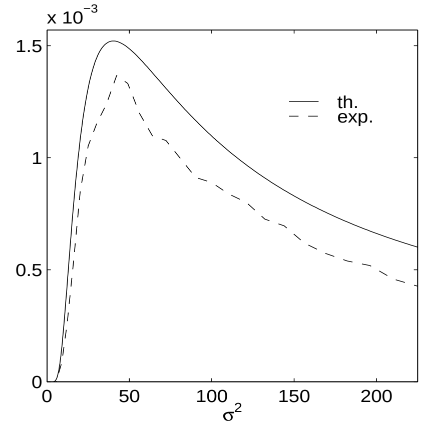

Figure 1: *1-d case: Theoretical and experimental SNR versus* σ^2. b_n *is Gaussian,* $c = 10$, $\lambda = .02$ *and* $\varepsilon = 1$.

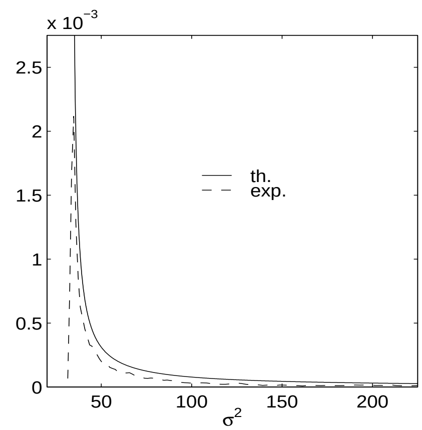

Figure 2: *1-d case: Theoretical and experimental SNR versus* σ^2. b_n *is uniform,* $c = 10$, $\lambda = .04$ *and* $\varepsilon = .25$.

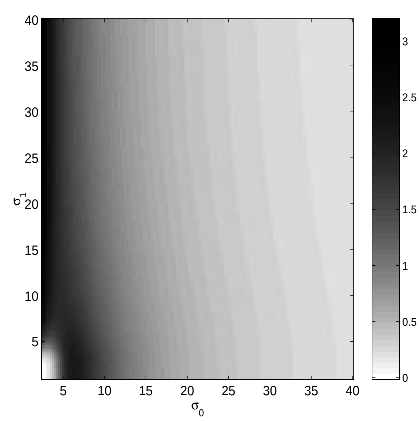

Figure 3: *2-d case:* $|\chi_0(\lambda_0)|$ *versus* σ_0 *and* σ_1. b_n *is Gaussian,* $c = 10$, $\alpha = 1.75$ *and* $\lambda = .02$.

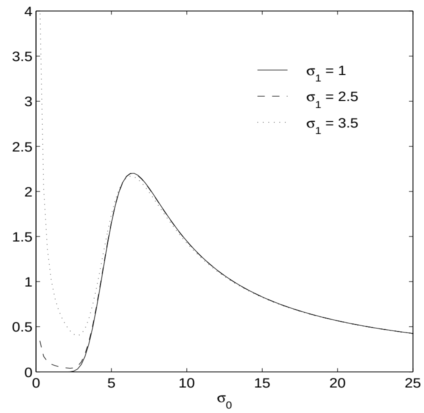

Figure 4: *2-d case:* $|\chi_0(\lambda_0)|$ *versus* σ_0 *for some fixed* σ_1. b_n *is Gaussian,* $c = 10$, $\alpha = 1.75$ *and* $\lambda = .02$.

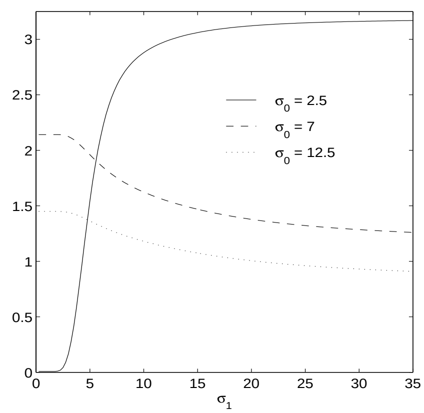

Figure 5: *2-d case:* $|\chi_0(\lambda_0)|$ *versus* σ_1 *for some fixed* σ_0. b_n *is Gaussian,* $c = 10$, $\alpha = 1.75$ *and* $\lambda = .02$.

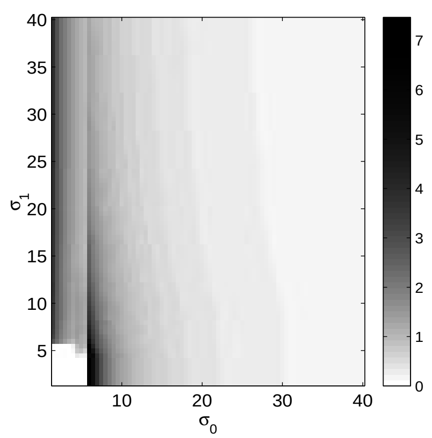

Figure 6: *2-d case:* $|\chi_0(\lambda_0)|$ *versus* σ_0 *and* σ_1. b_n *is Uniform,* $c = 10$, $\alpha = 1.75$ *and* $\lambda = .02$.

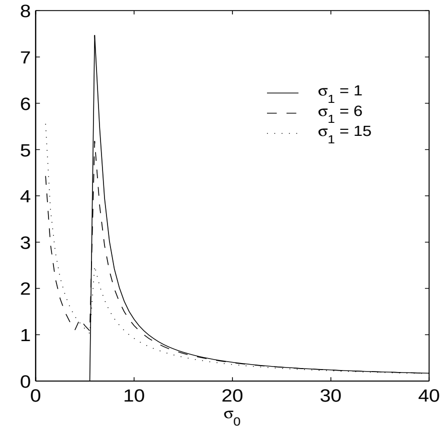

Figure 7: *2-d case:* $|\chi_0(\lambda_0)|$ *versus* σ_0 *for some fixed* σ_1. b_n *is Uniform,* $c = 10$, $\alpha = 1.75$ *and* $\lambda = .02$.

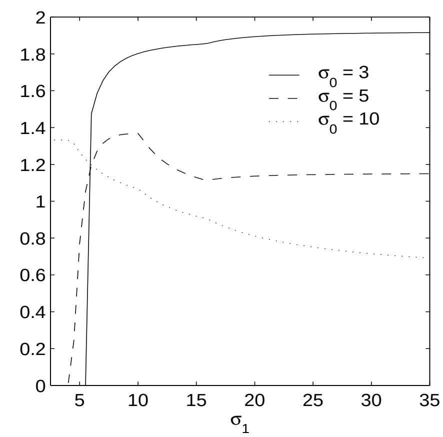

Figure 8: *2-d case:* $|\chi_0(\lambda_0)|$ *versus* σ_1 *for some fixed* σ_0. b_n *is Uniform,* $c = 10$, $\alpha = 1.75$ *and* $\lambda = .02$.

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