ANALYSIS OF CHAOTIC SIGNALS IN THE TIME-FREQUENCY PLANE

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ABSTRACT

In this paper we study chaotic signals generated from Chua's circuit: in particular we concentrate on the well known spiral attractor. Chaotic signals have a nonstationary nature, and hence the classical Fourier spectrum turns out to be inadequate for the analysis. Instead of it we use Time-Frequency Distributions, a powerful set of tools specifically designed for nonstationary signal analysis. Time-Frequency analysis shows that the considered chaotic signals are multicomponent, and can be thought as the sum of amplitude modulated (AM) signals with a constant carrier frequency. The basic components are extracted with a zero-phase distortion filtering, and the validity of the AM representation with constant carrier frequency is then verified. An approximation of the original chaotic signal is obtained summing up a limited number of its filtered components. The validity of this approach is verified by comparing the original chaotic attractor with that built through the approximated signals.

1. INTRODUCTION

In the last few years there have been significant advances in the study of chaotic systems and signals. It has been shown that chaotic signals and systems are useful to model several nonlinear phenomena occuring in physics, chemistry, biology and ecology (see [1]).

We consider the signals generated by a well known chaotic system: Chua's oscillator (see [2], [3]). This circuit, despite its simplicity, exhibits a large number of chaotic attractors [3]. As a case-study we restrict our attention to the well known spiral attractor (that can be observed in other chaotic systems, like Rossler's system).

Owing to the complex nature of a chaotic waveform, they have been mainly investigated through time-domain techniques: in fact the Fourier spectrum is not able to well represent the nonstationary nature of a chaotic signal.

We want to show that the Time-Frequency plane [4] is the correct domain to study chaotic signals. The application of Time-Frequency Distributions to chaotic systems has already been considered in [5] and [6], where a first order qualitative analysis is presented: however, the obtained results cannot be used to achieve an efficient representation of the chaotic signal. We will show that, since chaotic signals have a multicomponent nature, they can be thought as the sum of amplitude modulated one-component signals with a constant carrier frequency. Firstly these basic components are extracted through classical digital filtering techniques. The validity of the representation of such extracted components as constant carrier frequencies with amplitude modulation is tested. Then it is verified that few components yield an accurate representation of the original chaotic signal. Therefore, as a consequence result of such analysis is the possibility of reconstructing a chaotic signal by means of a simple model where few elementary modulated signals are generated and superimposed.

2. GENERATION OF CHAOTIC SIGNALS

The Chua's oscillator is a very simple circuit, made by two capacitors, one inductor, a nonlinear resistor and some linear resistors. The circuit can be described in the state space, taking as state variables the voltages on the two capacitors and the current through the inductor. The result is a system of three ordinary differential equations, that can be solved numerically for each set of initial conditions. In this work we generate the chaotic signals using a normalized version of this system of equations, defined as

$$\begin{cases} \frac{\mathrm{dx}}{\mathrm{dt}} = k\alpha(y - x - f(x)) \\ \frac{\mathrm{dy}}{\mathrm{dt}} = k(x - y + z) \\ \frac{\mathrm{dz}}{\mathrm{dt}} = k(-\beta y - \gamma z) \end{cases} \tag{1}$$

where

$$f(x) = m_1 x + \frac{1}{2}(m_0 - m_1)\{|x+1| - |x-1|\}$$

is the piecewise linear characteristic of the nonlinear resistor (other choices for $f(x)$ are still possible to observe chaos). The normalized state variables $x(t), y(t), z(t)$ are related respectively to the voltages of the two capacitors, and to the current through the inductor. Depending on the value of the parameters α, β, γ, the circuit is able to generate several chaotic attractors. We restrict our attention to the spiral

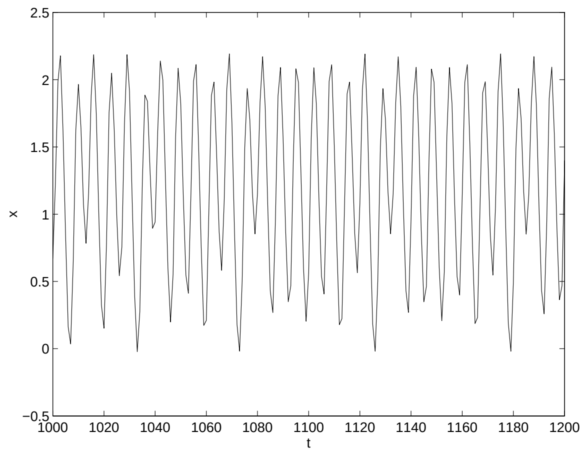

Figure 1: Signal $x(t)$ for the discrete time interval $t = [1000, 1200]$

attractor. To do this, we choose the following parameters:

$$\begin{array}{lll} \alpha = 8.50000425 & \beta = 14.2857143 & \gamma = 0 \\ m_0 = -1.142857144 & m_1 = -0.714285715 & k = 1 \end{array}$$

Every signal $x(t), y(t), z(t)$ is generated through the numerical algorithm described in [7] and is made of 8192 samples. In Fig. 1 the signal $x(t)$ is represented.

3. TIME-FREQUENCY ANALYSIS OF CHAOTIC SIGNALS

The classical spectrum analysis plays a crucial role in understanding the properties of a stationary signal. On the contrary, many problems arise in the interpretation of the spectrum of a nonstationary signal, althought it can be still correctly evaluated. The main problem is that the Fourier spectrum is not able to locate some isolated ”events” which occur in the temporal evolution of the signal itself. This is due to the periodic behaviour of the basis functions in the Fourier expansion.

3.1. Time-Frequency Analysis

Time-Frequency analysis [4], has been created to overcome these limitations. The basic idea is to map the 1D signal $s(t)$ under study in the so called Time-Frequency plane (TF plane), whose axis are precisely the time t and frequency f. The goal is to spread the instantaneous spectral information of the signal in this plane, making it possible to localize the various events occuring in the signal.

The mapping in the TF plane is called Time-Frequency Distribution (TFD) or Time-Frequency Representation (TFR). TFD can be linear, as for the Short-Time Fourier Transform (STFT) and the Continuous Wavelet Transform (CWT). They are called *bilinear* if the signal appears multiplied by itself, or generally of order n if its power of n is used. The bilinear class is also called *energetic class*, because the order 2 of the mapping implies that the TF plane is a representation of the energy of the signal in every (t, f) region. A subset of the bilinear class is the *Cohen class*, which collects the TFDs that are invariant under time-frequency shifts. The Wigner Distribution (WD) is the most important distribution in this class. It is defined as

$$W(t, w) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} s^* \left(t - \frac{\tau}{2}\right) s \left(t + \frac{\tau}{2}\right) e^{-j\tau\omega} \mathrm{d}\tau$$

where $\omega = 2\pi f$ and $s(t)$ is the analytic signal of the signal to analyze. Many others TFDs are designed starting from the WD. Despite of the great number of mathematical properties of the WD, it has two important limitations: it can be locally negative (and hence of difficult interpretation) and it is affected by Interference Terms (IT). IT are particularly present when analysing *multicomponent signals* (signals that can be seen as the sum of many basic one-component signals). This is due to the fact that the WD is bilinear, and hence the square of the sum of many signals produces many double product terms, precisely the IT.

Many TFDs have been designed to overcome these problems. In particular in this work we use the Smoothed Pseudo-Wigner Distribution (SPWD), defined as

$$S_{PWD}(t, \omega) = \iint g(t - t')H(w - w')W(t', w')\mathrm{d}t'\mathrm{d}w'$$

which performs a sophisticated 2D filtering of the WD using two smoothing windows g, H, able to reduce the IT. This is an important feature, because of the multicomponent nature of chaotic signals.

3.2. SPWD of Chaotic Signals

The classical Fourier analysis reveals that $x(t)$,$y(t)$,$z(t)$ are multicomponent signals, as observed in Fig. 2, where the power spectrum of $x(t)$ highlights the presence of a major energy concentration around a certain number of frequencies. The same behaviour can be observed by using a TFD analysis. However, if the signal is made of N components, the IT phenomenon arises. The number of IT is given by the Newton binomial $(N, 2) = N(N-1)/2$, and hence grows with $O(N^2)$. The choice of a filtered TFD able to reduce the IT becomes necessary. In this work we use a SPWD, with a Hanning window for both $g(t)$ and $H(w)$. In Fig. 3 the SPWD of the signal $x(t)$ is shown. The horizontal axis is the normalized frequency f, while the vertical axis represents the normalized time t. The frequency resolution

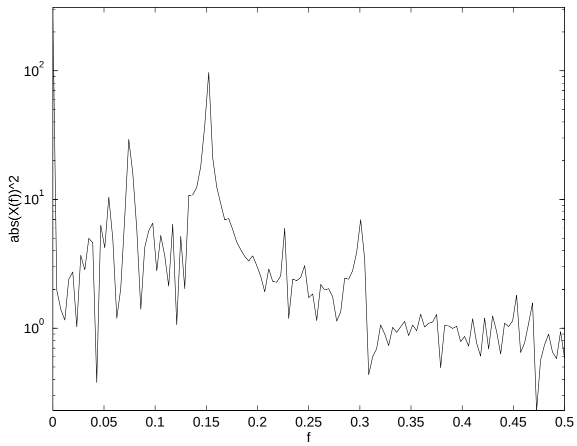

Figure 2: Power spectrum of the signal $x(t)$ shown if Fig. 1. The horizontal axis shows the normalized frequency. The vertical axis is in logarithmic scale

is 1/256. The visualization software draws a slice for every discrete time value t: every slice represents the SPWD evaluated at time t on the 256 points of the frequency axis.

3.3. Multicomponent Representation

The SPWD confirms that $x(t)$ is a multicomponent signal. In Fig. 3, it is possible to recognize four major energy concentrations $x_1(t), \ldots, x_4(t)$, located around the normalized frequency values $f_1 = 0, f_2 = 0.075, f_3 = 0.15, f_4 = 0.3$. (the same result holds also for $y(t), z(t)$). Every component is concentrated around its central frequency f_i, and exhibits a time-varying spread about it. This spread can be expressed as the local variance $\sigma_{\omega|t}$, and it can be linked to the concept of instantaneous bandwith (see [4]), defined as

$$B_t = \left| \frac{A'(t)}{A(t)} \right| \tag{2}$$

where $A(t)$ is the instantaneous amplitude defined as the modulus of the analytic signal, and $A'(t)$ denotes its derivative. The spread in frequency is clearly time-dependent for every component, and this means that the local bandwidth is a time-varying function. Due to the time-varying nature of B_t, from (2) we infer that also $A(t)$ must be a time-varying function, and hence a simple model for the i-th component of the type

$$x_i(t) = A_i(t) e^{j\omega_i t} \tag{3}$$

can be introduced. Notice that this representation cannot be inferred if only the Fourier spectrum is considered. This is due to the fact that the power spectrum shows how the total power of the signal is divided on every frequency component for the *entire duration* of the signal, and is not able to

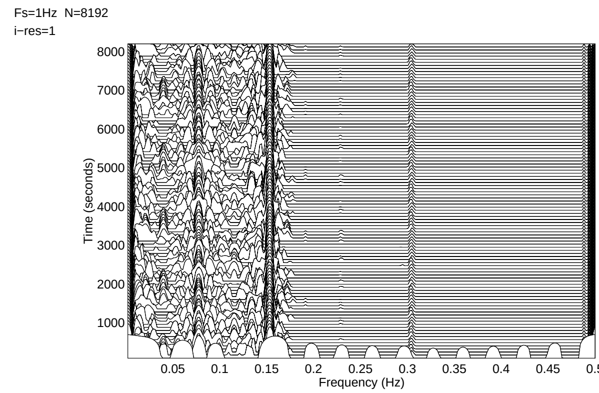

Figure 3: SPWD of signal $x(t)$. The z axis is in logarithmic scale

localize and hence represent the time variation of every instantaneous component. TF analysis turns out to be a crucial tool to localize and correctly represent this kind of instantaneous spectral properties.

Another interesting point is that chaotic signals produced in the spiral attractor case can be seen as the limit of an infinite sequence of period-doubling bifurcations. The bifurcation process is obtained by monotonically varying a parameter (for example α), called the bifurcation parameter. The interest in this fact is that if the bifurcation process is stopped before reaching the chaotic region, we always observe a signal made of a finite number of frequencies, and hence with a power spectrum made of a finite number of spectral lines. But in the chaotic region, the TF plane shows that a good signal representation is the sum of the one-component signals modelled as in (3). The next natural step is to extract these one-component signals, with the goals to

- check if they are really made of a carrier frequency with amplitude modulation, as inferred from the TFD;
- build an approximation of the original chaotic signals by direct summation of these components, and test the validity of the reconstruction.

4. COMPONENT EXTRACTION

The extraction of the basic one-component signals of every state variable has been done by digital filtering. The extraction has been limited to the four highest energy components of each chaotic signal. We have designed the four digital filters looking at the TF plane obtained for $x(t)$, but the same filters have been applied to $y(t)$ and $z(t)$. The used filter is

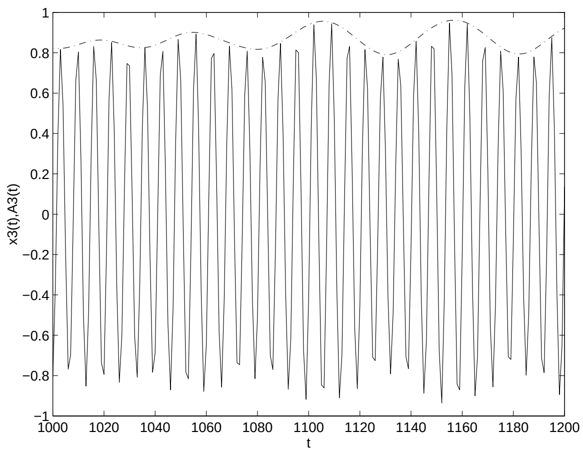

Figure 4: Continuous line: filtered component $x_3(t)$. Dotted line: instantaneous amplitude $A_3(t)$ obtained from the analytic signal of $x_3(t)$

a Butterworth bandpass filter, with its center frequency coincident with the carrier frequency, to exploit its maximally flat behaviour for reducing amplitude distortion.

Care must be taken with the phase distorsion introduced by the filter. In fact the time-varying spectral properties of the chaotic signals (well represented in the TF plane) are related to the instantaneous phase of the signal: therefore it is very important to reduce the phase distorsion to the minimum. This has been done with a zero-phase distortion filtering, by filtering the signal, reversing the output, filtering and reversing again. The resulting output is a signal with the same phase of the input. In Fig. 4 we show the filtered output $x_3(t)$. The dotted line represents the amplitude $A_3(t)$ obtained computing the modulus of the analytic signal of $x_3(t)$. In Sect. 5 we justify the representation of the filtered component as a constant carrier frequency with amplitude modulation.

5. VALIDATION OF THE AM REPRESENTATION

We want to check if the filtered one-component signals are really made of a constant carrier frequency with amplitude modulation. To do this we consider as an example the extracted component $x_3(t)$, and we evaluate its instantaneous phase $\varphi_3(t)$ as the phase of its analytic signal. With $A_3(t)$ we still represent the modulus of the analytic signal as in Sect. 4. Since in general the instantaneous frequency $f_i(t)$ is related to the phase in a differential way [4]

$$f_i(t) = \frac{1}{2\pi}\varphi'(t)$$

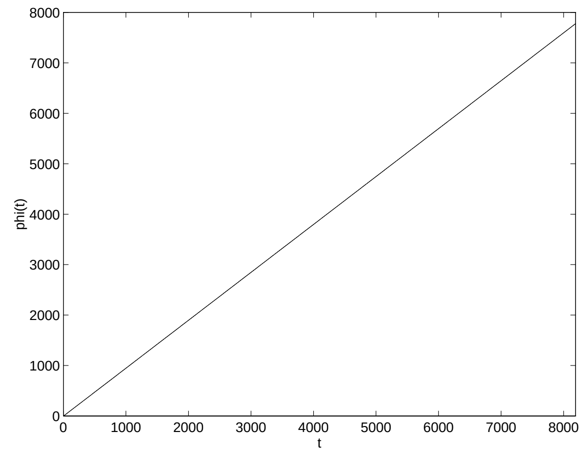

Figure 5: Instantaneous phase of the analytic signal computed for the filtered signal $x_3(t)$.

and $f_3(t)$ has to be constant, then $\varphi_3(t)$ must be a linear function of t. In Fig. 5 the computed phase $\varphi_3(t)$ is shown, and it is possible to notice that it presents a very good qualitative linear behaviour. To be sure of the linearity of $\varphi_3(t)$, we build a synthetic signal defined as

$$s_3(t) = A_3(t)e^{j\hat{\varphi}_3(t)}$$

forcing its phase to be a linear function of t

$$\hat{\varphi}_3(t) = a * t + b$$

The parameters a, b are then evaluated with a least square minimization with respect to the target phase $\varphi_3(t)$. In Fig. 6 the signal $x_3(t)$ and the approximation $s_3(t)$ are compared on a small time interval. It is possible to notice the validity of the AM representation.

6. RECONSTRUCTION OF THE SPIRAL ATTRACTOR

We show that the original chaotic signals $x(t)$,$y(t)$,$z(t)$ can be accurately approximated by signals $\tilde{x}(t)$,$\tilde{y}(t)$,$\tilde{z}(t)$, obtained by summing a small number of filtered components, extracted as indicated in the previous section.

In the present example a good reconstruction is obtained by using only four components. For example, $\tilde{x}(t)$ turns out to be represented as

$$\tilde{x}(t) = \sum_{i=1}^{4} x_i(t) = \sum_{i=1}^{4} A_i(t)e^{j\omega_i t}$$

The accuracy of the approximation is verified by comparing the phase state representation of the original and of the reconstructed signal. The result is shown in Fig. 7, where it is possible to appreciate the quality of the reconstruction.

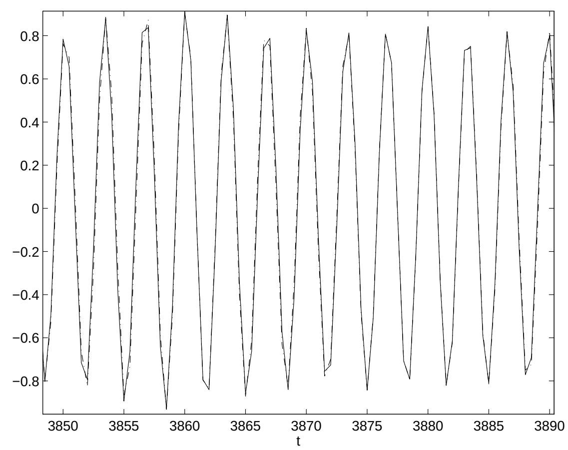

Figure 6: Comparison between the filtered component $x_3(t)$ (continuous line) and the linear phase synthetic signal $s_3(t)$ (dotted line).

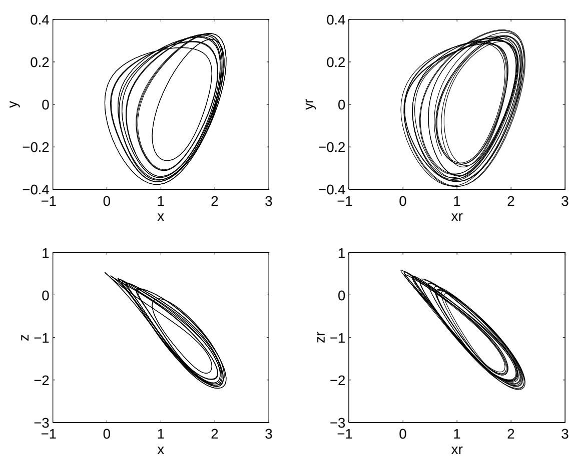

Figure 7: Phase planes of the original and of the reconstructed attractor. In the left column the original phase planes (x, y) and (x, z) of the spiral attractor are represented, while in the right column the corresponding reconstructed versions are shown

7. CONCLUSIONS

In this paper we have analyzed chaotic signals coming from Chua's circuit when its parameters are chosen to lead to a spiral attractor. The study has been done by Time-Frequency Distributions, a powerful set of tools specifically designed for the analysis of nonstationary signals.

The representation in the Time-Frequency plane has shown that such chaotic signals are multicomponent. In particular they can be thougth as the sum of amplitude modulated one-component signals with a constant carrier frequency. With a zero-phase distortion filtering we have extracted these components and checked the prevision made on their nature. Then we have shown that by summing a small number of these components, a good approximation of the original chaotic signal may be obtained.

8. REFERENCES

[1] Nicolis G., Prigogine I., *Exploring Complexity*, New York: W. H. Freeman & Co., 1989

[2] Special Issue on ”Chaos in nonlinear electronic circuits”, *IEEE Trans. Circuits Syst.*, vol. 40, parts I and II, no. 10, and part I no. 11, 1993

[3] Special Issue on ”Chua's circuit: A paradigm for chaos”, *Int. J. Circuits, Syst., Computers*, vol. 3, no. 1, 2, Mar-June 1993

[4] Cohen L., *Time-Frequency Analysis*, Prentice Hall, NJ, 1995

[5] Ping C., ”Study of chaotic dynamical systems via time-frequency analysis”, *Proceedings of the IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis*, IEEE, New York, NY, USA; 1994; p.357-60

[6] Adamopoulos P.G., Hammond J.K., Lee J.S., ”The Wigner distribution, multi-equilibria nonlinear systems and chaotic behaviour”, *ICASSP 88*, IEEE, New York, NY, USA; 1988; p.2216-19 vol.4

[7] Biey M., Chialina S., Hasler M., Premoli A., ”Piecewise-linear analysis for Chua's circuit family, including the computation of Lyapunov exponents”, in [3], p. 793-804