STOCHASTIC CALCULUS NUMERICAL EVALUATION OF CHAOTIC COMMUNICATION SYSTEM PERFORMANCE

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ABSTRACT

Based on stochastic calculus, we provide a rigorous formulation for the numerical evaluation of the error probabilities of two modulation techniques for the chaotic Lorenz communication system with AWGN disturbance. These results provide further understanding on the robust synchronization ability of the Lorenz system to noise. The synchronization robustness of Lorenz systems for various time scaling factors is also discussed. An approximate model of the variance of the sufficient statistic of the chaotic communication is derived, which permits a comparison of the chaotic communication system performance to a conventional communication system.

1. INTRODUCTION

Since Pecora and Carroll [1] have theoretically and experimentally shown that two identical chaotic dynamic systems can be synchronized, chaotic signals with inherently broadband, noise-like and unpredictable properties have been proposed for possible communication modulation/coding waveforms [2][3]. A chaotic system possess self-synchronization property if it can be decomposed into two subsystems: a drive system and a conditionally stable response subsystem that synchronize when driven with a common signal. Based on the drive-response configuration of synchronization, specific chaotic communication techniques, such as chaotic signal masking, chaotic switching modulation [2][3][4], and dynamic feedback modulation of information signal [5], have been considered. As the basic operation of those techniques is highly dependent on the self-synchronization property of the chaotic systems, the robust ability of the systems to perturbation in the drive signal is a crucial factor in determining the system performance. Since the AWGN channel constitutes the most basic component of a communication link, the understanding of the robust self-synchronization ability of a chaotic communication system to WGN is necessary for system design. However, because of the inherent nonlinearity of a chaotic system, it is generally difficult to obtain an analytic solution and thus numerical simulations are needed.

Cuomo *et al* [2] have addressed the numerical evaluation of the SNR improvement of the Lorenz chaotic communication system based on deterministic numerical algorithms. It is known that commercial numerical computational packages using the standard Euler or Runge-Kutta(RK) algorithms for deterministic differential equations to approximate the solution to nonlinear stochastic differential equation(SDE) will incur significant errors [6]. This is particularly true for a nonlinear chaotic dynamical system modeling the transmitter inputting into an AWGN channel and followed by another nonlinear chaotic dynamical system.

In this paper, we use the stochastic calculus approach to perform the integration algorithm for the sample functions of nonlinear dynamic systems excited by the stochastic white noise. Depending on the precise interpretation of the white noise, there are two different solutions to the SDE based on the Stratonovich or Ito integral [9]. Using the conversion between them, a correct numerical integration algorithm in the Ito sense is introduced. With this algorithm, the correct error probability of the robust self-synchronization Lorenz communication system with AWGN perturbation is presented. The self-synchronization robustness of Lorenz systems for various time scaling factors, changing the speed of system evolution, is also discussed. Furthermore, we explicitly demonstrate the performance evaluation of this model using deterministic numerical algorithms yields incorrect results. Finally, the numerical evaluation and comparison of error probabilities among dynamic feedback and chaotic switching modulation for a Lorenz system and a conventional communication system are provided.

2. PROBLEM DESCRIPTION

The modified Lorenz system [2] is given by

$$dx/d\tau \quad = \quad \sigma(y(t) - x(t))$$

$$\begin{aligned} dy/d\tau &= rx(t) - y(t) - 20x(t)z(t) \\ dz/d\tau &= 5x(t)y(t) - bz(t), \end{aligned} \quad (1)$$

where σ, r, and b are system parameters, and $\tau = t/K$, in which K is a time scaling factor. As shown in Pecora and Carroll , and Cuomo *et. al*, two Lorenz systems in drive-response structure, illustrated in Figure 1 , can be synchronized in the absence of perturbation in the drive signal. To characterize the robust ability of synchronization to white noise, the modified Lorenz system can be interpreted as the drive system, the signal $v(t)$ is the received waveform at the response system as defined by

$$\begin{aligned} dx_r/d\tau &= \sigma(y_r(t) - x_r(t)) \\ dy_r/d\tau &= rv(t) - y_r(t) - 20v(t)z_r(t) \\ dz_r/d\tau &= 5v(t)y_r(t) - bz_r(t), \end{aligned} \quad (2)$$

where $v(t) = x(t) + n(t)$, and $n(t)$ is white Gaussian noise with zero mean and power spectrum density σ_n^2. The chosen coefficients are $\sigma = 16$, $r = 45.6$, and $b = 4$.

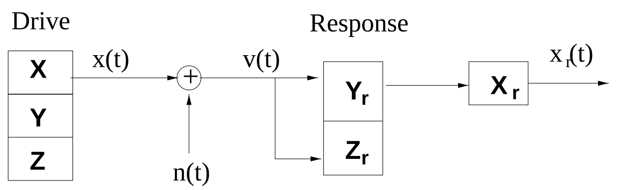

Figure 1: Drive-response synchronization schematic diagram.

We define the vector $\mathbf{X} = [x, y, z, x_r, y_r, z_r]^T$. The entire system composed of the drive subsystem and the response subsystem can be viewed as a nonlinear system with an external white noise input, and has the following standard form $\dot{\mathbf{X}}_i = f_i(\mathbf{X}) + g_i(\mathbf{X})n(t),\ i = 1, 2, \ldots, 6$, where $n(t)$ is the WGN with zero mean and unity variance. The evolution of the above equation is then given by

$$\mathbf{X}_i(t_0 + h) = \mathbf{X}_i(t_0) + \int_{t_0}^{t_0+h} f_i(\mathbf{X})dt + \int_{t_0}^{t_0+h} g_i(\mathbf{X})n(t)dt. \quad (3)$$

A commonly made mistake is to treat the third term of (3) as deterministic ordinary calculus and apply the standard Euler or RK integration algorithm. That is, if $g_i(\mathbf{X})$ is smooth function, the integration result can be approximated as

$$\int_{t_0}^{t_0+h} g_i(\mathbf{X})n(t)dt \approx g_i(\mathbf{X}(t_0))Y_1 h, \quad (4)$$

where Y_1 is a Gaussian random variable with zero mean and unity variance.

In order to illustrate this issue clearly, we use the above algorithm to characterize the robust self-synchronization ability of a Lorenz system by numerical computation. To compare the results of Cuomo [2], the time scale of the Lorenz system is taken with the factor of $K = 2505$. The simulation results are computed numerically for 10 seconds and the first few seconds of data are discarded to eliminate initial transient effects. The simulation results are shown in Fig. 2, one of which is consistent with that in Cuomo, *et al.* The definitions of the input SNR and output SNR quantities in Fig. 2, are defined as

$$\begin{aligned} Input\ SNR &= 10\log_{10}(\sigma_x^2/\sigma_n^2) \\ Output\ SNR &= 10\log_{10}(\sigma_x^2/\sigma_e^2), \end{aligned}$$

where σ_x^2 is the power of transmitted signal $x(t)$, and σ_e^2 is the power of the synchronization error $e(t) = x(t) - x_r(t)$. Then the output SNR varies with the integration step size. Furthermore, the output SNR decreases by 3 dB as the step size is doubled. This is not a reasonable consequence for a given system which is excited by a stationary external white noise.

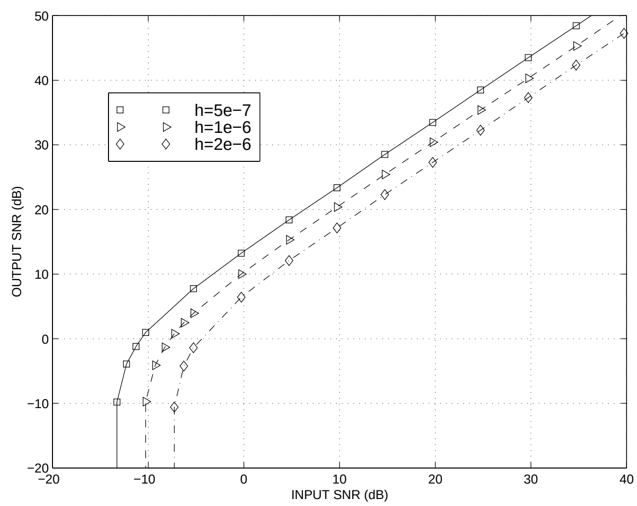

Figure 2: Synchronization performance using standard Runge-Kutta method($K = 2505$).

3. NUMERICAL ALGORITHM FOR SDE

For the above-described chaotic system, the corresponding SDE in the sense of Ito is

$$dX_i = f_i(\mathbf{X})dt + g_i(\mathbf{X})dw(t),\ \mathbf{X}(t_0) = \mathbf{X}_0, \quad (5)$$

where $w(t)$ is the one-dimensional Wiener process or Browian motion, and $\mathbf{X}_0$ is initial conditions.

A simple and numerically realizable mean square approximation of the random variable $\mathbf{X}(t_0 + T)$ in Ito sense is given by Milshtein [7]. Mannella and Palleschi [8] have derived a more accurate integration algorithm in the sense of Stratonovich [9], which treats the white noise as the limiting behavior of bandlimited white noise, and the approximate results are summarized as the following algorithm

$$X_i(h) - X_i(0) = \delta X_i^{1/2} + \delta X_i^1 + \delta X_i^{3/2} + \delta X_i^2 + \dots, \quad (6)$$

where h is the integration step size, and

$$\delta X_i^{1/2} = g_i \int_0^h dw(t) = \sqrt{h} g_i Y_1, \quad (7)$$

where Y_1 is a Gaussian random variable with zero mean and unity variance, while the remaining terms are given in equation(6) of [8].

According to Stratonovich [9], the integral in the sense of Stratonovich can be converted into an Ito integral by adding one correction term. That is, if the SDE is modified and is given in the sense of Stratonovich as

$$dX_i = \left[f_i(\mathbf{X}) - \frac{1}{2} \sum_J \frac{\partial g_i(\mathbf{X})}{\partial \mathbf{X}_J} g_J(\mathbf{X}) \right] dt + g_i(\mathbf{X}) dw(t), \quad (8)$$

the system evolution of equation(8) by using the above numerical algorithm is statistically equivalent to the evolution of equation(5) in the sense of Ito [10], which is desired here because the stochastic term $n(t)$ is true white noise. We use this algorithm to re-simulate the robust self-synchronization ability to white noise and the simulation results are shown Fig. 3. Clearly, simulation results are consistent with different integration step sizes, which is necessary consequence. We also note that the robust self-synchronization ability of the Lorenz system to white noise dropped dramatically compared to Fig. 2.

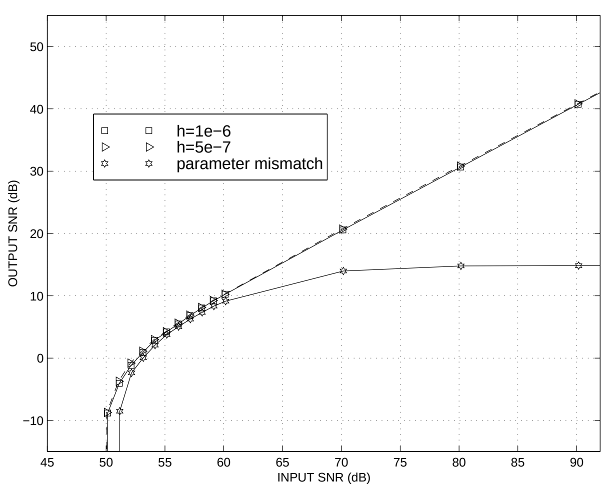

Figure 3: Synchronization performance using Ito / Mannella and Palleschi method($K = 2505$).

According to Cuomo *et al.*[2], the time scaling factor K in the Lorenz system equation is used to speed up the evolution of the Lorenz system. That is, the Lorenz system with a time scaling factor greater than one will synchronize within the shorter transient period, which is needed in practical high-speed communication systems. This technique is useful if synchronization between drive and response systems is still preserved. In fact, however, this time scaling factor will enhance the noise effect on synchronization. To see this effect, we provide the robust self-synchronization ability for the Lorenz system with time scaling factors $K = 100, 25,$ and 1, as shown in Figure 4. The simulation results are computed for $30, 100,$ and 1000 seconds and the first $15, 30,$ and 100 seconds of transient data are discarded, respectively. It is clear that synchronization performance degrades by $10 \log_{10} K$ dB as the time scaling factor K is used, and this results in shorter transient period.

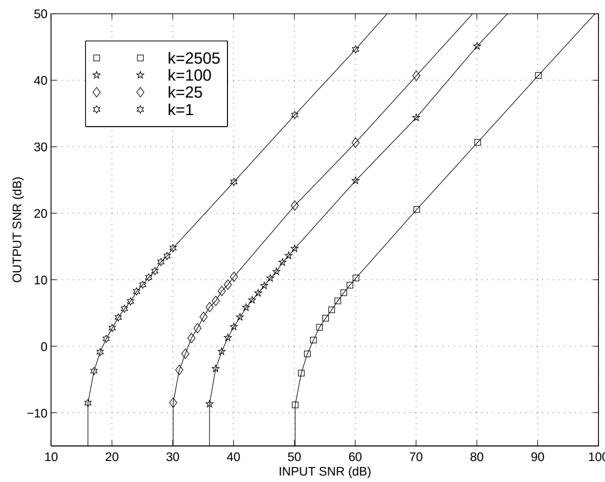

Figure 4: Synchronization performance for various time scaling factors using Ito / Mannella and Palleschi method

4. CHAOTIC COMMUNICATIONS

Two popular communication techniques, chaotic signal masking and chaotic switching modulation(CSM) have been proposed by [2] and [3] independently. The improved masking algorithm by Milanovic [5], which is called dynamic feedback modulation(DFM), has been shown theoretically more robust than the chaotic masking scheme. However, quantitative comparison among those methods in the sense of digital communication performance have not been seen. In this paper, we present the DFM and CSM communication systems performance.

4.1. Chaotic Switching Modulation

The basic idea of a CSM communication system, shown in Figure 5, is to encode the binary data $m(t)$ with different chaotic attractors by modulating the transmitter parameters and to transmits chaotic drive signal $x_m(t)$. At the receiver, the parameter modulation will produce a synchronization error between the received drive signal and the regenerated drive signal with an error amplitude that depends on the modulation. Using the synchronization error, the binary data can be detected.

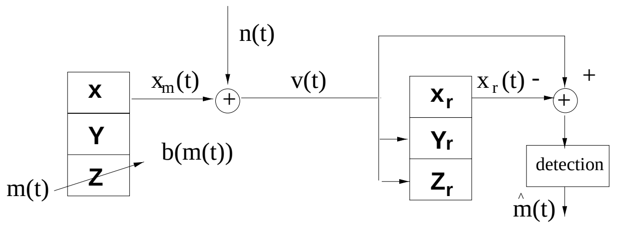

Figure 5: Chaotic switching modulation communication system.

To be specific, the parameter b of the transmitter is modulated by the binary data $m(t)$. The synchronization error $e_s(t)$ is defined as the difference between noisy received signal $r(t) = x_m(t)+n(t)$ and the regenerated signal $x_r(t)$ at the receiver. For this binary hypothesis problem, the sufficient statistic η_s is the power of the synchronization error signal after discarding some transient data and is defined by

$$\eta_s = \frac{1}{T}\int_{t_0}^{t_0+T} e_s^2(t)dt. \tag{9}$$

4.2. Dynamic Feedback Modulation

In the DFM communication system shown in Figure 6, the transmitted signal $v(t) = x(t) + m(t)$, the combination of information signal $m(t)$ and chaotic signal $x(t)$, is communicated to the receiver which is identical to the chaotic transmitter. Since the reconstructed signal $x_r(t)$ will be identical to $x(t)$ in the absence of noise $n(t)$, the information signal $m(t)$ can be decoded from the received signal by using

$$\hat{m}(t) = x(t) + m(t) - x_r(t). \tag{10}$$

This analog communication technique can be applied to the binary data communication by setting $m(t) = A$ if the binary information data is one, and $m(t) = -A$ if the binary data is zero. The sufficient statistic η_d of detection is the average of the error signal at the receiver after discarding the initial transient period of synchronization, and is given as

$$\eta_d = 1/T\int_{t_0}^{t_0+T} e_d(t)dt, \tag{11}$$

where the error signal $e_d(t)$ is defined as $e_d(t) = v(t) + n(t) - x_r(t)$. Since the feedback information will affect the chaotic property, the information level A should be chosen

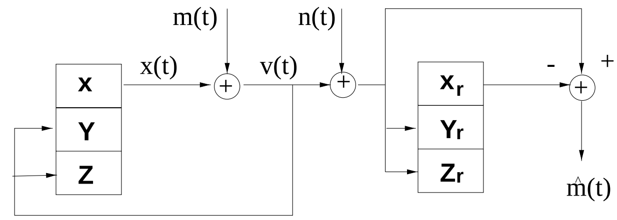

Figure 6: Dynamic feedback modulation communication system.

carefully to make transmitter still chaotic to maintain the communication security.

Two set of parameters $P_1 = \{\sigma = 16, r = 45.6, b = 4\}$, and $P_{-1} = \{\sigma = 16, r = 45.6, b = 5\}$ are used for the CSM communication system at the transmitter while the receiver uses the P_1 parameter set of [2]. For the DFM communication system, information level $A = 0.2$ is chosen. Simulation results of error probabilities of these two chaotic communication systems are shown in Fig. 7. It is clear that DFM is better than CSM because the error probability of 0.5 for CSM with SNRs below 60dB is expected from Fig. 3.

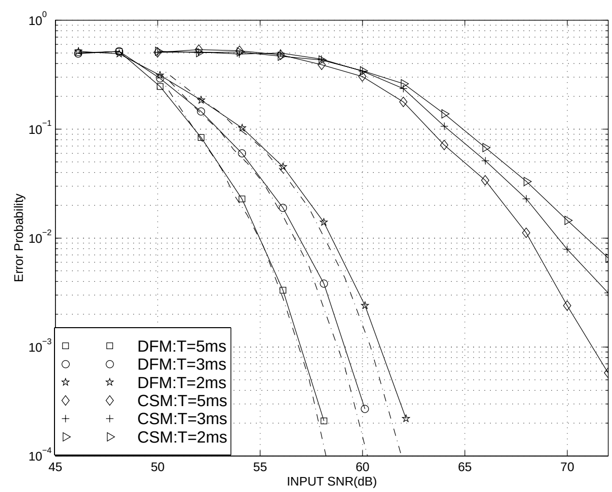

Figure 7: Chaotic communication system performance for DFM and CSM ($K = 2505$).

Note that the definition of SNR in Fig. 7 is different from the usual one used by the communication community. For the DFM chaotic communication system, in addition to the channel white noise, synchronization errors between the transmitter and receiver also cause errors in detection. Moreover, we found that the variance of η_d can be approximately modeled by

$$var(\eta_d) = c(N_0)N_0/2T = c(N_0)var(\eta_{BPAM}) \tag{12}$$

if $T > \tau_c$, where $\tau_c = 0.1$ms is the correlation time of synchronization error with time scaling factor $K = 2505$, and

$c(N_0)$ is shown in Fig. 8. With this approximate model, we can estimate the bit error probability of the DFM communication system using

$$P_e = Q\left(\sqrt{2E_m/c(N_0)N_0}\right), \qquad (13)$$

where E_m is the energy of the information signal $m(t)$. The results are shown in Fig. 7 as dashed lines, which are consistent with the brute force simulation results.

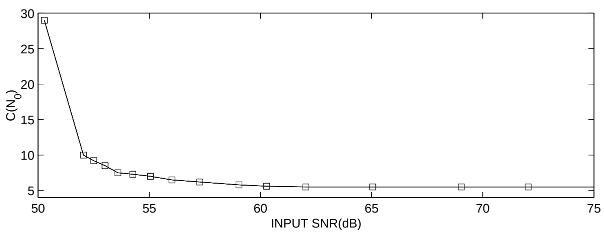

Figure 8: Coefficient $c(N_0)$.

To compare fairly with the BPAM communication system, we need to exclude the effect of synchronization error as much as possible in evaluating the bit error probability of the DFM communication system. From Fig. 8, we know that $c(N_0) = 5.5$ is independent of the channel noise strength if the input SNR is greater than 62dB, which means the bit error probability depends only on the ratio E_b/N_0 if the observation period $T > \tau_c$, where E_b is the energy of the transmitted signal. Therefore, the DFM communication performance is worse than BPAM by

$$10\log_{10}(5.5(P_x + P_m)/P_m) = 23.5dB \qquad (14)$$

in the usual SNR definition, where $P_x = 1.6$ and $P_m = 0.04$ are the average powers of the chaotic signal $x(t)$ and information signal $m(t)$, respectively.

5. CONCLUSION

We have provided a rigorous formulation for numerical evaluation of error probabilities of two modulation techniques for chaotic Lorenz communication system with AWGN disturbance. Based on analysis of self-synchronization errors, an approximate model of the variance of sufficient statistic of chaotic communication is derived, which permits an estimation of the chaotic communication system performance and a comparison to a conventional communication system.

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