Analysis of Self Noise in a Clock Recovery Systems With a High-Order Nonlinearity

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ABSTRACT: This paper presents a new technique to compute efficiently the I and Q spectra, and the $I-Q$ cross-spectrum of the self noise appearing at the output of the zero-memory, high order nonlinear device employed in a clock recovery system. It is known that these spectra play an important role in the phase jitter performance of the clock regenerator. The results are very general and applicable to many cases of practical interest. A numerical example provided at the end of the paper shows that the new approach yields very accurate results and is much faster that the usual computer simulation method.

1. Introduction

A popular method for clock synchronization in a coherent communication system consists of passing the incoming signal, either at IF or at baseband, through a zero-memory device with an even nonlinearity and then feeding the resulting waveform to a phase locked-loop or equivalently to a bandpass filter centered at the pulse repetition frequency, $1/T$, so that a discrete tone is generated at the symbol rate frequency as shown in Fig.1. Many forms of nonlinearities may be used for this purpose. The most common are square-law [1], absolute value [2], and fourth law [3]. In a high signal-to- noise case, the recovered clock is mainly contaminated by a data dependent noise , which is called *self noise* or *pattern noise*. Gardner has pointed out that the self noise component that is in phase with the recovered clock plays a different role from the component that is quadrature [4]. To properly analyze the contribution of each of these noise components to phase jitter, the power spectra of the two noise components must be found separately.

Dealing with a square-law clock regenerator for PAM systems, the I and Q self-noise power spectra was obtained analytically in [5]. When dealing with strongly band limited signaling pulses, nonlinearities other than the square-law must be considered [6,8]. Unfortunately, clock circuits implemented with nonsquare-law devices are hardly tractable mathematically [9] and, in fact, their phase jitter performance has only been evaluated by computer simulation [10]. In this paper, we present a new technique to compute the I and Q spectra, and the $I-Q$ cross spectrum of the self noise appearing at the output of the zero-memory device with any even, high-order nonlinearity. The technique is based on the iterative computation of the high-order cross-moments of the input signal to the clock recovery circuit. The results are rather general and are applicable to many cases of practical interest.

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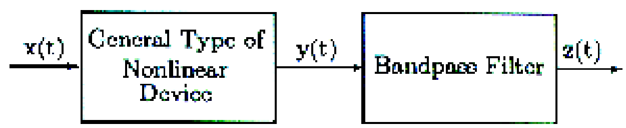

Fig.1 Block diagram of a clock recovery system

2. Problem Statement and System Model

The demodulated baseband signal at the input of the clock regenerator is represented by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k g(t-kT), \tag{1}$$

where the data sequence $\{a_k\}$ is assumed to be a statistically independent binary sequence taking the values ± 1, with equal probabilities and $g(t)$ is the pulse shape employed in transmission. The high order zero-memory nonlinear device employed in the clock regenerator is defined by a finite power series of the form:

$$y(t) = f[x(t)] = \sum_{n=0}^{N} c_n x^{2n}(t), \tag{2}$$

where the c_n's, $n=0,1,\ldots,N$ are given real constants and $2N$ is the order of the nonlinearity. The nonlinear(NL) device is followed by a phase locked-loop or, equivalently, by a narrowband bandpass filter whose transfer function $H(f)$ is centered at the symbol rate frequency $1/T$. The output $y(t)$ of the NL device consists of a periodic component $\overline{y}(t)$ with period T, plus a zero mean random part $N(t)$ called self noise.

$$\begin{aligned} \overline{y}(t) &= E[y(t)] \\ N(t) &= y(t) - E[y(t)]. \end{aligned} \tag{3}$$

The narrow-band output filter will pass the desired periodic component $E[y(t)]$ at clock frequency $1/T$. The transfer function of the output filter is assumed to satisfies the bandlimiting condition: $H(f) = 0 \quad \text{for} \quad ||f| - \frac{1}{T}| > \frac{1}{2T}$. Hence, as far as this filter is concerned, $y(t)$ can be written as

$$y(t) = (2P)^{1/2}\cos(2\pi t/T + \theta) - N(t)$$

where,

$$(P/2)^{1/2}\exp j\theta = < E[y(t)]\exp(-j2\pi t/T >_t$$

and $< f(s,t)_s$ denotes the time average of $t(s,t)$ with respect to s.

Our main objective is to decompose the self noise component $N(t)$ in (3) into N_I, in-phase and N_Q, the quadrature components and to present an efficient method to determine their power spectra from which the phase jitter of the timing wave can be determined. $N(t)$ can be decomposed into I and Q components with respect to $\cos(2\pi t/T + \theta)$ as follows .

$$N_I(t) = N(t)\cos(2\pi t/T + \theta) + \tilde{N}(t)\sin(2\pi t/T + \theta) \quad (4)$$
$$N_Q(t) = \tilde{N}(t)\cos(2\pi t/T + \theta) - N(t)\sin(2\pi t/T + \theta) \quad (5)$$

where $\hat{N}(t)$ denotes the Hilbert transform of $N(t)$.

3. Power Spectral Densities of Decomposed Self Noise

Since $x(t)$ is a cyclostationary process(CT) so is $N(t)$ in (3). Thus its correlation function expressed as $R_N(t+\tau/2, t-\tau/2)$, which is a function of two independent variables, t and τ, is periodic in t with period T for each value of τ. Its Fourier series expansion and its time dependent complex Fourier coefficients $r_N^k(\tau)$ are by given (6) and (7), respectively.

$$R_N(t+\tau/2, t-\tau/2) = \sum_{k=-\infty}^{+\infty} r_N^k(\tau)\exp(2\pi kt/T), \quad (6)$$

$$r_N^k(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} R_N(t+\tau/2, t-\tau/2) \times \exp(-2\pi kt/T). \quad (7)$$

As will be seen shortly, the time dependent complex Fourier coefficient defined by (7) plays an important role in determining the power spectral densities of I and Q components of the self-noise. Gardner calls the $r_N^k(\tau)$ as the "cyclic autocorrelation function" and its Fourier transform denoted by $S_N^k(f)$ as the "cyclic spectral density" [11].

We now concentrate on the exact computation of the power spectral densities of I and Q and the $I - Q$ cross spectrum of self noise. Since $N(t)$ is a CS process, so are $N_I(t)$ and $N_I(t)$. Therefore, their autocorrelation function are also periodic with period T and, they can be expanded into Fourier series as follows:

$$R_{N_I}(t+\tau/2, t-\tau/2) = \sum_{k=-\infty}^{+\infty} r_{N_I}^k(\tau)\exp(2\pi kt/T), \quad (8)$$

$$R_{N_Q}(t+\tau/2, t-\tau/2) = \sum_{k=-\infty}^{+\infty} r_{N_Q}^k(\tau)\exp(2\pi kt/T). \quad (9)$$

Furthermore, the $N_I(t)$ and $N_Q(t)$ are correlated and their correlation plays an important role in the phase jitter performance of the clock regenerator. Similarly, the cross-correlation of $N_I(t)$ and $N_Q(t)$,$R_{N_I N_Q}(t+\tau/2, t-\tau/2)$ is also T-periodic and can be expressed in Fourier series as

$$R_{N_I N_Q}(t+\tau/2, t-\tau/2) = \sum_{k=-\infty}^{+\infty} r_{N_I N_Q}^k(\tau)\exp(2\pi kt/T). \quad (10)$$

The I and Q power spectra and the $I-Q$ cross power spectrum denoted by $P_I(f), P_Q(f)$ and $P_{IQ}(f)$ can be obtained from (8) (9) and (10) by taking first the time averages over T and then taking the Fourier transforms of the resulting functions. That is,

$$\begin{aligned} P_I(f) &= F\{< R_{N_I}(t+\tau/2, t-\tau/2) >_t\} \\ &= F\{r_{N_I}^0(\tau)\} = S_{N_I}^0(f) \end{aligned} \quad (11)$$

$$\begin{aligned} Q_I(f) &= F\{< R_{N_Q}(t+\tau/2, t-\tau/2) >_t\} \\ &= F\{r_{N_Q}^0(\tau)\} = S_{N_Q}^0(f) \end{aligned} \quad (12)$$

$$\begin{aligned} P_{IQ}(f) &= F\{< R_{N_I N_Q}(t+\tau/2, t-\tau/2) >_t\} \\ &= F\{r_{N_I N_Q}^0(\tau)\} = S_{N_I N_Q}^0(\tau) \end{aligned} \quad (13)$$

where "F" denotes the Fourier transform operation.

We now present exact analytical expression to calculate the I and Q and $I-Q$ cross power spectral densities of the self noise $N(t)$, generated at the output of any general type of zero-memory non-linear device. The derivation is lengthy and therefore is given in the Appendix. The final expressions are as follows:

$$\begin{aligned} P_I(f) = \; & S_N^0(f+1/T)u(f+1/T) \\ & + S_N^0(f-1/T)u(-f+1/T) \\ & + 2Re\{S_N^2(f)\}e^{-j2\theta}u(1/T-|f|) \end{aligned} \quad (14)$$

$$\begin{aligned} P_Q(f) = \; & S_N^0(f+1/T)u(f+1/T) \\ & + S_N^0(f-1/T)u(-f+1/T) \\ & - 2Re\{S_N^2(f)\}e^{-j2\theta}u(1/T-|f|) \end{aligned} \quad (15)$$

$$\begin{aligned} P_{IQ}(f) = \; & j\,[S_N^0(f+1/T)u(f+1/T) \\ & - S_N^0(f-1/T)u(-f+1/T) \\ & -2Im\{S_N^2(f)\}e^{-j2\theta}u(1/T-|f|)] \end{aligned} \quad (16)$$

where $u(.)$ denotes the unit step function and the $S_N^k(.), k = 0, 2$ is the cyclic spectrum of $N(t)$, that is $F\{r_N^k(\tau)\}$. $Re\{.\}$ and $Im\{.\}$ denote the real and imaginary parts of a complex quantity.

4. Iterative Computation of the Power Spectral Densities

The equations for $P_I(f), P_Q(f)$ and $P_{IQ}(f)$ in (14-16) are the main results of this paper. The main steps to compute them iteratively are summarized below as follows.

Step 1. Determine the autocorrelation function $R_N(t,s)$ of the self noise, iteratively as follows:

From Eqs.(3) and (2), the autocorrelation function $R_N(t,s)$ can be expressed as

$$\begin{aligned} R_N(t,s) &= \sum_{m=0}^{N}\sum_{n=0}^{N} c_m c_n \{E[x^{2m}(t)x^{2n}(s)] \\ &\quad -E[x^{2m}(t)]E[x^{2n}(s)]\}. \end{aligned} \tag{17}$$

which is a linear combination of the high-order moments and high-order cross moments of $x(t)$ at time instants t and s, namely,

$$\begin{aligned} M_{2m,2n}(t,s) &= E[x^{2m}(t)x^{2n}(s)], \\ M_{2n}(t) &= E[x^{2n}(t)]. \end{aligned}$$

We now present the expressions to compute M_{2n} and $M_{2m,2n}$, iteratively. The derivations are lengthy and involved. The details can be found in [12]. The final iterative expressions are as follows:

Iterative Computation of M_{2n}

$$M_{2n} = -\left\{\sum_{i=1}^{n}\binom{2n-1}{2i-1}(-1)^i M_{2(n-i)} f^{2i-1}\right\}, \tag{18}$$

where $M_0 = 1$, and

$$f^{2i-1} = \begin{cases} \sum_k g^2(\alpha - kT) & \text{if } i = 1 \\ \frac{2^i(2^i-1)}{2i}|B_{2i}|\sum_{l=-\infty}^{\infty}[g(\alpha - lT)]^{2i} & \text{if } i = 2,3,\ldots \end{cases}$$

where $g(\cdot)$ is the pulse shape as defined in (1) and B_i's are the Bernoulli numbers.

Iterative Computation of $M_{2m,2n}$

For $n = 1,2,\ldots$ and $m = 0,1,2,\ldots$,

$$M_{2m,2n} = -\sum_{i=0}^{n-1}\left[\binom{2n-1}{2i}P_{2i} + \binom{2n-1}{2i+1}P_{2i+1}\right] \tag{19}$$

where

$$\begin{aligned} P_{2i} &= \sum_{j=0}^{m}\binom{2m}{2j}(-1)^{m+n-i-j} \\ &\quad \times f_2^{2m-2j,2n-1-2i} M_{2j,2i} \end{aligned}$$

$$\begin{aligned} P_{2i+1} &= \sum_{j=1}^{m}\binom{2m}{2j-1}(-1)^{m+n-i-j} \\ &\quad \times f_2^{2m-2j+1,2n-2-2i} M_{2j-1,2i+1}. \end{aligned}$$

Similarly, for $n = 0,1,2,\ldots$ and $m = 0,1,2,\ldots$,

$$M_{2m+1,2n+1} = -\left[\sum_{i=0}^{n}\binom{2n}{2i}Q_{2i} + \sum_{i=0}^{n-1}\binom{2n}{2i+1}Q_{2i+1}\right] \tag{20}$$

where

$$\begin{aligned} Q_{2i} &= \sum_{j=0}^{m}\binom{2m+1}{2j}(-1)^{m+n+1-i-j} \\ &\quad \times f_2^{2m-2j,2n-1-2i} M_{2j,2i} \end{aligned}$$

$$\begin{aligned} Q_{2i+1} &= \sum_{j=0}^{m}\binom{2m+1}{2j+1}(-1)^{m+n-i-j} \\ &\quad \times f_2^{2m-2j,2n-2i-1} M_{2j+1,2i+1} \end{aligned}$$

and, initially, for $n = 0$ and $m = 0,1,2,\ldots$,

$$M_{2m,0} = -\sum_{j=1}^{m}\binom{2m-1}{2j-1}(-1)^j f_1^{2j-1,0} M_{2(m-j),0}. \tag{21}$$

Note that $f_1^{p,q}$ and $f_2^{p,q}$ are defined as follows:

$$f_1^{p,q} = \begin{cases} \sum_k u_k^2, & \text{if } p=1, q=0 \\ \sum_k u_k v_k, & \text{if } p=0, q=1 \\ C(p,q)|B_{p+q+1}|\sum_k u_k^{p+1}v_k^q, & \text{if } p+q \in \text{odd } \aleph \\ 0, & \text{if } p+q \in \text{even} \aleph. \end{cases}$$

$$f_2^{p,q} = \begin{cases} \sum_k v_k^2, & \text{if } p=0, q=1 \\ \sum_k u_k v_k, & \text{if } p=1, q=0 \\ C(p,q)|B_{p+q+1}|\sum_k u_k^{p}v_k^{q+1}, & \text{if } p+q \in \text{odd } \aleph \\ 0, & \text{if } p+q \in \text{even} \aleph. \end{cases}$$

where $\aleph$ denotes the set of natural numbers and

$$\begin{aligned} u_k &\triangleq g(t-kT), \\ v_k &\triangleq g(s-kT), \\ C(p,q) &\triangleq \frac{2^{p+q+1}(2^{p+q+1}-1)}{p+q+1} \end{aligned}$$

Step 2: Determine the complex Fourier coefficients $r_N^k, k = 0, 2$ by expanding $R_N(t,s)$ into Fourier series as follows:

$$\begin{aligned} R_N(t+\tau/2, t-\tau/2) &= \sum_{k=-\infty}^{+\infty} r_N^k(\tau)\exp(2\pi kt/T), \\ r_N^k(\tau) &= \frac{1}{T}\int_{-T/2}^{T/2} R_N(t+\tau/2, t-\tau/2) \\ &\quad \times \exp(-2\pi kt/T). \end{aligned}$$

and finally,

Step 3: Find $S_N^k, k = 0, 2$ by taking the fast Fourier transforms of $r_N^k(\tau)$ and use them in the power spectral densities expressions, given by Eqs.(14-16).

5. A Numerical Example

We now compute the I, Q spectra and $I-Q$ cross spectrum of the self noise at the output of a fourth order nonlinear device when the input is a baseband signal represented by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k g(t - kT), \qquad (22)$$

where data sequence a_k is assumed to be a statistically independent binary sequence taking values $\pm$, with equal probabilities and $g(t)$ is the Raised-Cosine type of pulse shape with excess bandwidth factor α. We computed the power spectra of each of the components of the Self Noise for several excess bandwidth factors α and the results are summarized in Figs.2 and 3. We observe from these curves that both spectra increase monotonically with α for $0.1 < f < 0.8$. Also note that, for the Nyquist pulse shape used here, the cross-power spectrum was very small and not distinguishable from computational noise. We also compared the results obtained here with Fang's results[8] and concluded that they are in a very good agreement with each other.

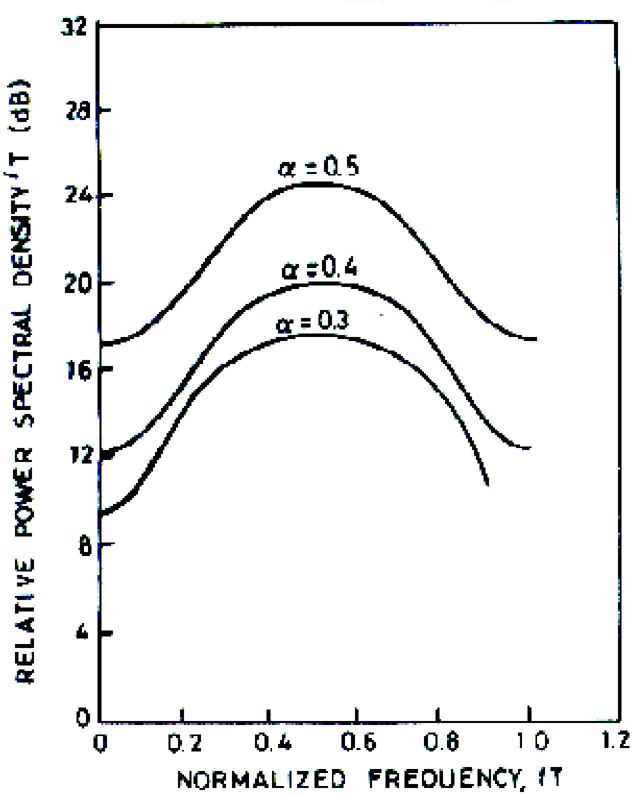

Fig.2 I- Spectrum for Nyquist pulse

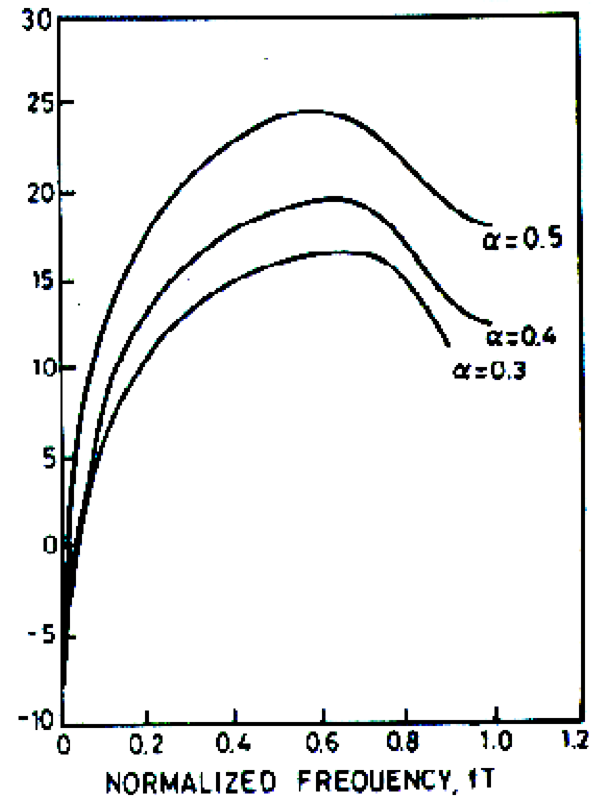

Fig.3 Q- Spectrum for Nyquist pulse

6. Conclusions

Equations for $P_I(f), P_Q(f)$ and P_{IQ} obtained in Sec.3 are the main results of this paper. The main steps to compute them are summarized as follows:

- Determine $R_N(t, s)$, iteratively
- Determine the complex Fourier coefficients $r_N^k(\tau), k = 0, 2$ by expanding $R_N(t, s)$ into Fourier series.
- Find $S_N^k(.), k = 0, 2$ by taking the fast Fourier transforms of $r_N^k(\tau)$ and use them in the expressions for $P_I(f), P_Q(f)$ and P_{IQ}.

A numerical example given in Sec.5 showed that this new approach yields results which are in a very good agreement with those given in the literature and is computationally much faster than the usual computer simulation approach.

Appendix

Derivations of I, Q Power Spectra and $I-Q$ Cross Power Spectrum of the Self Noise

Derive (14-16) as follows: Consider the complex envelope

$$\begin{aligned} \Gamma(t) &= \left[N(t) + j\hat{N}(t)\right] e^{-j(2\pi t/T+\theta)} \\ &= N_I(t) + jN_Q(t). \end{aligned} \tag{.1}$$

Solve this equation to obtain

$$\begin{aligned} N(t) &= \frac{1}{2}\Gamma(t)e^{j(2\pi t/T+\theta)} + \frac{1}{2}\Gamma^*(t)e^{-j(2\pi t/T+\theta)} \\ N_I(t) &= \frac{1}{2}\Gamma(t) + \frac{1}{2}\Gamma^*(t) \\ N_Q(t) &= \frac{1}{2j}\Gamma(t) - \frac{1}{2j}\Gamma^*(t). \end{aligned}$$

Then, we can easily show that

$$\begin{aligned} S_N^k(f) &= \frac{1}{4}\left[S_\Gamma^k(f-1/T) + S_{\Gamma^*}^k(f+1/T)\right. \\ &\quad \left. + S_{\Gamma\Gamma^*}^{k-2}(f) + S_{\Gamma^*\Gamma}^{k+2}(f)\right] \end{aligned} \tag{.2}$$

where $S_N^k(f), S_\Gamma^k(f)$ are the cyclic spectra of $N(t)$ and $\Gamma(t)$, respectively, and $S_{\Gamma\Gamma^*}^k(f)$ is the cross cyclic spectrum of $\Gamma(t)$ and $\Gamma^*(t)$, as defined in Sec.3. Using the cyclic spectra of Hilbert transforms given in [10, page 406],

$$S_{\hat{N}}^k(f) = \begin{cases} -S_N^k(f), & \text{if, } |f| < |k|/2T \\ +S_N^k(f), & \text{if, } |f| > |k|/2T \end{cases} \tag{.3}$$

$$S_{N\hat{N}}^k(f) = S_{\hat{N}N}^k(-f) = \begin{cases} -jS_N^k(f), & \text{if, } f < k/2T \\ +jS_N^k(f), & \text{if, } f > k/2T \end{cases} \tag{.4}$$

we obtain the following

$$S_\Gamma^k(f) = S_N^k(f+1/T)u(f+1/T-|k|/2T), \tag{.5}$$

$$S_{\Gamma\Gamma^*}^k(f) = S_N^{k+2}(f)e^{-j2\theta}u(k/2T+1/T-|f|), \tag{.6}$$

$$\begin{aligned} S_{N_I}^k(f) &= \frac{1}{4}\left[S_\Gamma^k(f) + S_{\Gamma^*}^k(f)\right. \\ &\quad \left. + S_{\Gamma\Gamma^*}^k(f) + S_{\Gamma^*\Gamma}^k(f)\right], \end{aligned} \tag{.7}$$

$$\begin{aligned} S_{N_Q}^k(f) &= \frac{1}{4}\left[S_\Gamma^k(f) + S_{\Gamma^*}^k(f)\right. \\ &\quad \left. - S_{\Gamma\Gamma^*}^k(f) - S_{\Gamma^*\Gamma}^k(f)\right], \end{aligned} \tag{.8}$$

$$\begin{aligned} S_{N_I N_Q}^k(f) &= \frac{j}{4}\left[S_\Gamma^k(f) - S_{\Gamma^*}^k(f)\right. \\ &\quad \left. - S_{\Gamma\Gamma^*}^k(f) + S_{\Gamma^*\Gamma}^k(f)\right]. \end{aligned} \tag{.9}$$

Finally, substituting (A.5) and (A.6) into (A.7) to (A.9) with $k = 0$ one obtains (14) to (16).

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