EQUALIZATION OF NONLINEAR TRANSMISSION CHANNELS : ALGEBRAIC SYSTEMS THEORY APPROACH

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ABSTRACT

In this paper, we introduce a new algebraic framework, issued from the theory of nonlinear system, for studying the problem of equalization of nonlinear transmission channels. For this purpose, we introduce the nonlinear system rank concept. Our method gives the condition on the rank of the nonlinear system, here the transmission channel, which permits the perfect equalization. We show that, for a nonlinear satellite transmission system, this operation is possible except in some domain of singularity that can be determined. In this case, the equalization will be not possible even by a nonlinear equalizer.

Keywords : nonlinear system, difference algebra, rank of nonlinear system, left invertibility, equalization.

1. INTRODUCTION

The problem of equalization is certainly one of the major research topics in telecommunication. Its purpose is the elimination of channel induced degradation effects. In the linear case, this problem has been intensively studied for several years [1] [2]. Most of available methods, however, are proposed for scalar systems (single input-single output).

In the other hand, the equalization of nonlinear systems is of considerable practical interest, although it has attracted much less attention. Many real-life systems exhibit nonlinear characteristics. Since high speed data transmission in digital radio systems and in satellite communication uses amplifier devices which usually work near saturation. These amplifier devices introduce memoryless nonlinearities which, combined with the effects of the transmission and reception filters, become nonlinearities with memory. The input/output relationship of the channel becomes nonlinear and is usually modeled by polynomial or by Volterra filter of finite order [3] [4].

Recent works have showed that the systems theory promises interesting applications in the equalization of linear transmission channels [5] [6] [7]. The aim of this work is to revisit the problem of equalization of nonlinear transmission channels. We apply, here, the tools developed in the framework of difference algebra for left invertibility.

The organization of this paper is as follows. The mathematical tools are first presented in section 2. In section 3, we present a short overview on the theory of nonlinear systems, and in particular, the input output invertibility. The link between equalization and left invertibility is established in section 4. Section 5 is devoted to the application of these tools for the equalization of satellite channels modeled by an input-output nonlinear system.

2. SUMMARY OF DIFFERENCE ALGEBRA

All the fields considered here are of characteristic zero. The usual fields of rational, real and complex numbers are well known examples of such fields.

2.1. A *difference field* is a commutative field K which is equipped with a monomorphism

$$\delta \ : K \rightarrow K \tag{1}$$

called *transformation*, which satisfies the following rules :

$$\begin{aligned} \forall a, b \in K, \quad \delta(a+b) &= \delta(a) + \delta(b), \\ \delta(ab) &= \delta(a).\delta(b), \\ \delta(a) = 0 &\Leftrightarrow a = 0. \end{aligned} \tag{2}$$

In the context of signal processing, δ should be interpreted as a backward shift of one unit of time, i.e.,

$$\delta\left(x(t)\right) = x(t \Leftrightarrow 1). \tag{3}$$

A constant is an element $c \in K$ such that $\delta(c) = c$.

2.2. A difference field extension L/K is given by two difference fields K, L such that :

a) $K \subset L$;

b) the transformation of K is the restriction to K of the transformation of L.

2.3. An element a of L is said to be *transformally algebraic* over K if, and only if, it satisfies an algebraic difference equation with coefficients in K. It means that there exists a polynomial $P(\epsilon_0, \cdots, \epsilon_\nu)$ over K such that $P(a, \delta(a), \cdots, \delta^\nu(a)) = 0$; where $\delta^\nu(a) = a(t \Leftrightarrow \nu)$. The element a is said to be *transformally transcendental* over K if, and only if, it is not transformally algebraic over K.

2.4. A set $\{ai/i \in I\}$ of elements of L is said to be transformally K-algebraically independent if, and only if, the set $\left\{\delta^{(j)}(a_i)\, / i \in I, j \in lN\right\}$ is K-algebraically independent. Such an independent set, which is maximal with respect to inclusion, is called a transformal transcendence basic of L/K. Two such bases have the same cardinality which is called the transformal transcendence degree of L/K; we denote it by *transf tr* $d^\circ L/K$. If there exists a finite set of difference quantities $\omega = (\omega_1, \omega_2 \cdots, \omega_s)$, such that L is generated by K and ω, we say that L/K is a finitely generated difference field extension. Set $L = K < \omega >$.

2.5. Denote by $K[\delta]$ the set of polynomial linear difference operators $\sum a_i \delta^i$ $(a_i \in K)$. A *difference k-module* M is a left $K[\delta]$-module [8].

If there exists a finite set of difference quantities $\omega = (\omega_1, \omega_2, \cdots, \omega_s)$, such that M is generated by ω, we say that M is a finitely generated difference module. Set $M = [\omega]$.

2.6. A *difference filtration* of a difference module M is given by an ascending chain (M_r) of K-vector spaces in M.

For example,

$$M_r = span\,\{\omega(n), \omega(n \Leftrightarrow 1), \cdots, \omega(n \Leftrightarrow r)\} \qquad (4)$$

is a difference filtration of $M = [\omega]$.

3. INVERTIBILITY OF DISCRETE TIME SYSTEMS

Input-output invertibility is a most classic subject of investigation in control systems. A vast literature has been devoted to it, which will be partly analyzed in the sequel. Left (resp. right) invertibility is equivalent to the possibility of recovering the input from the output (resp. independence of the output variable).

For a constant linear system, the invertibility is obvious by the use of the transfer matrix. In fact, the system is right invertible (left invertible) if and only if the transfer matrix is right invertible (left invertible). In the case where the system is described by a state representation, such a concept remains applicable. In addition, we dispose of a structure algorithm due to Silverman [9].

In the nonlinear case, the classical methods have tried to generalize, under different aspect, the structure algorithm [10]. Thanks to difference algebra, introduced by Fliess [11][12], that the problem has been solved for the first time. Thus, the rank of a linear and nonlinear system, called transformal output rank has been defined for the first time. This rank generalizes the rank of the transfer matrix. In the following of this paragraph, we will give a short review of the main results, issued from the difference algebra approach, on the inversion of discrete time systems.

Linear system : Let (S) be a given linear system, with the input $u = (u_1, u_2, \cdots, u_m)$ and the output $y = (y_1, y_2, \cdots, y_p)$. The *transformal output rank* ρ of (S) is the rank of the difference module $[y]$ [11][12]. The system (S) is left (resp. right) invertible if, and only if, $\rho = m$ (resp. $\rho = p$)

Proposition 1 : [13][14] For r big enough

$$\rho = dim\,[V_{r+1}] \Leftrightarrow dim\,[V_r] \qquad (5)$$

where $V_r = span\,\{y(n), y(n \Leftrightarrow 1), \cdots, y(n \Leftrightarrow r)\}$.

Nonlinear system : Let (Σ) be a given nonlinear system, with the input $u = (u_1, u_2, \cdots, u_m)$ and the output $y = (y_1, y_2, \cdots, y_p)$. The transformal output rank ρ of (Σ), is the transformal transcendence degree of $K < y > /K$ [11][12]. The system (Σ) is left (resp. right) invertible if, and only if, $\rho = m$ (resp. $\rho = p$)

Proposition 2 : [13][14] For r big enough

$$\rho = dim\,[V_{r+1}] \Leftrightarrow dim\,[V_r]\,, \qquad (6)$$

where $V_r = span\,\{dy(n), dy(n \Leftrightarrow 1), \cdots, dy(n \Leftrightarrow r)\}$, and $dy(n)$ is the differentials of $y(n)$.

4. EQUALIZATION AND LEFT INVERTIBILITY

During the transmission of digital message, through a channel linking two points in the space, the information is contained in a single composed of symbols. The physical channel (cable, wire, free space, etc.), introduces distortion in the transmitted signal. For example, due to the limited frequency band of the channel, the received signal is distorted by the intersymbol interference.

The equalization is a technique employed to recover the transmitted signal from the received one, and this in spite of the channel induced interference. Its role is then to inverse the effects of the channel. The equalization is then closely related to the left invertibility problem of input-output system.

Proposition 3 : Let C be a transmission channel with m the number of sources and p the number of sensors. The condition that guarantees the existence of an equalizer is that the rank of C, considered as input-output system, is equal to m [15].

5. APPLICATION TO THE EQUALIZATION OF A SATELLITE TRANSMISSION SYSTEM

5.1. Problem formulation

Let's apply this proposition to a satellite transmission system equalization. The transmission satellite channel (figure 6), considered here, is described by the following nonlinear input-output difference equations :

$$\begin{cases} s_n & = a_n + h_1 a_{n-1}, \\ \omega_n & = \zeta s_n + \xi s_n^3, \\ y_n & = \omega_n + h_2 \omega_{n-1}; \end{cases} \tag{7}$$

where h_1, h_2, ζ and $\xi \in I\!R$, a_n is the emitted message and y_n the received message.

The study of the invertibility of this channel will be done by using the following filtration $V_r = span\{dy_n, dy_{n-1}, \cdots, dy_{n-r}\}$. The differentials of s_n, ω_n and so of y_n are given by :

$$\begin{cases} ds_n & = da_n + h_1 da_{n-1}, \\ d\omega_n & = \left(\zeta + 3\xi s_n^2\right) ds_n, \\ dy_n & = d\omega_n + h_2 d\omega_{n-1}. \end{cases} \tag{8}$$

One has :

$$dy_n = \alpha_n da_n + \beta_n da_{n-1} + \lambda_n da_{n-2} \tag{9}$$

where :

$$\begin{cases} \alpha_n & = \zeta + 3\xi s_n^2, \\ \beta_n & = h_1 \alpha_n + h_2 \alpha_{n-1}, \\ \lambda_n & = h_1 h_2 \alpha_{n-1}. \end{cases} \tag{10}$$

Then $dim\left[span\{dy_n\}\right] = 1$, $dim\left[span\{dy_n, dy_{n-1}\}\right] = 2$ and $dim\left[span\{dy_n, dy_{n-1}, \cdots, dy_{n-r}\}\right] = r + 1$. We deduce that for $r \in I\!N$:

$$dim\left[V_{r+1}\right] \Leftrightarrow dim\left[V_r\right] = 1. \tag{11}$$

Then the system is left invertible and the equalization of such channel is possible when this rank condition remain valid.

5.2. Simulation and results

The rank of the system described by the equation (7) is equal to 1 (which corresponds to the condition announced above) except on a hypersurface at which $dim\left[V_{r+1}\right] \Leftrightarrow dim\left[V_r\right] = 0$; this situation occurs when $\alpha_n = 0$. Then the system is left invertible, and so the equalization of such channel is possible, except on a hypersurface defined by :

$$\left\{(a, b) \in I\!R^2 / \zeta + 3\xi s_n^2 = 0\right\} \tag{12}$$

To illustrate this paragraph, we consider an equalizer using a Volterra filter. Figure 2 exhibits the Mean Square Error (MSE) evolution, corresponding to the difference between the equalizer output and the transmitted data. Two cases of channels are considered. For the first case (figure 2-a), where the values of parameters a and b does not verify the condition (12), the equalizer succeeds to recover the correct transmitted data (The MSE take low values). For the second case (figure 2-b) corresponding to the condition (12), the MSE converge to the infinity and the equalizer does not succeed to invert correctly the channel.

6. CONCLUSION

In this paper, we have proposed to use the tools developed in the framework of difference algebra to study the left invertibility of nonlinear transmission channels. We have used the concept of the transformal output rank of a nonlinear system to establish the condition of left invertibility and so the possibility of equalization of the channel. The condition is verified on a satellite transmission channel which is modeled by a nonlinear system.

It will be interesting to find, by using this tools of algebraic systems theory, the equivalent of the concept of minimum and non-minimum phase channels in the nonlinear context. Which is of great benefit for blind equalization.

7. REFERENCES

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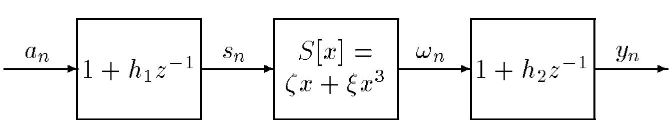

Figure 1: Nonlinear channel transmission modelling

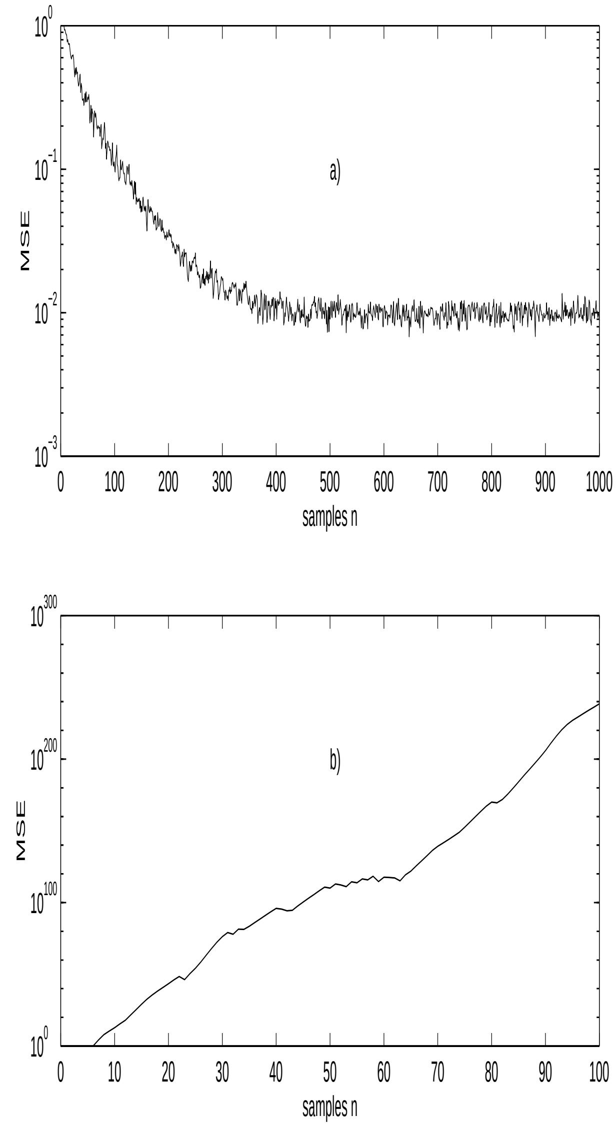

Figure 2: The equalizer MSE evolution : equalization of a satellite channel (case **a)** : $\zeta = \Leftrightarrow 0.375$ and $\xi = 0.25$, case **b)** : $\zeta = \Leftrightarrow 0.375$ and $\xi = 0.5$)