BIDIMENSIONAL α -STABLE MODELS WITH LONG-RANGE DEPENDENCE

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ABSTRACT

In this paper, we propose an α -stable 2D extension of the 1D fractional Gaussian noise. The considered model exhibits long-range dependence properties, while its stability index allows us to control the degree of spikyness of the synthesized fields. A least square method for the estimation of the three parameters of this model is also described and its performances are evaluated by a Monte Carlo study.

1. INTRODUCTION

Stable processes have turned out to be good models for many impulsive signals and noises, when the probability distributions of the highly variable data have "heavy" tails. These distributions have infinite variance and undefined higher-order moments, but it was pointed out in [8] that many signal processing algorithms based on second-order statistics can be transposed to fractional lowerorder moments.

In the context of linear modeling, the main advantage of stable distributions is to allow us to define non-Gaussian processes whose probability laws are easily deduced from those of their driving noise. In this work, we will be interested in the design of a specific class of 2D discrete-space processes with stable distributions. The considered Jean-Christophe Pesquet

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models have long range dependence properties and, consequently, they could provide interesting alternatives to image modeling techniques based on the 2D fractional Brownian motion.

The outline of this paper is as follows. In Section 2 we recall some useful facts on stable distributions. Conditions for the existence of linear 2D stable processes are then given in Section 3. Section 4 introduces a new model for stable fields which, in some sense, is the bidimensional equivalent of 1D FARIMA(0, d, 0) stable processes. In Section 5, we present a simple method for identifying this model and, in Section 6, we provide simulation examples to illustrate the effectiveness of this approach.

2. STABLE DISTRIBUTIONS

The family of stable distributions [7] is interesting, since linear transforms preserve the distribution of any linear combination of independent, identically distributed α -stable random variables. This property is directly related to the definition of stable random variables: a random vriable has a stable law if for all positive numbers A and B, there exists a positive number C and a real number D such that

$$AX_1 + BX_2 \stackrel{d}{=} CX + D,$$

where X_1 and X_2 are independent random variables with the same distribution as X.

Another fundamental result is the generalized central limit theorem [5], which represents an alternative definition of a stable law: a random variable X has a stable law if and only if there exists a sequence of independently, identically distributed random variables $(Y_i)_{i \in \mathbb{N}}$ and sequences of positive numbers $(d_n)_{n \in \mathbb{Z}}$ and real numbers $(a_n)_{n \in \mathbb{Z}}$ such that

$$\frac{Y_1 + Y_2 + \ldots + Y_n}{d_n} + a_n \stackrel{d}{\Rightarrow} X,$$

where the symbol $\stackrel{d}{\Rightarrow}$ denotes the convergence in distribution.

There does not exist explicit forms for most of the probability densities of stable variables. Exceptions are the Gaussian distribution ($\alpha = 2$), the Cauchy distribution ($\alpha = 1$) and the Lévy distribution ($\alpha = 1/2$). Hence, an important way to characterize an α -stable law is by means of its characteristic function which has the following form, in the symmetric case:

$$\mathbf{E}\left\{e^{i\theta X}\right\} = \exp\left(-\sigma^{\alpha}|\theta|^{\alpha}\right)$$

where $\alpha \in (0, 2]$ and $\sigma \ge 0$ are the two parameters of the symmetric α -stable (S α S) law.

We note $X \sim S_{\alpha}(\sigma, 0, 0)$. The parameters α and σ of this distribution are called, respectively, the index of stability and the scale parameter. For Gaussian random variable ($\alpha = 2$), σ is proportional to the standard deviation while the mean is zero. When $\alpha > 1$, the scale parameter induces a norm $||X||_{\alpha} = \sigma$ on a vector space of jointly S α S random variables.

One of the major difficulties encountered when considering stable random variables is the impossibility of defining the variance and, when $\alpha \leq 1$, even the mean. More precisely, if we have a random variable $X \sim S_{\alpha}(\sigma, 0, 0)$, where $0 < \alpha < 2$, then

$$\begin{split} & \mathbb{E}|X|^p < \infty \quad \text{as} \quad 0 < p < \alpha, \\ & \mathbb{E}|X|^p = \infty \quad \text{as} \quad p \ge \alpha. \end{split}$$

3. CONDITIONS FOR THE EXISTENCE OF LINEAR STABLE RANDOM FIELDS

In the sequel, we will be concerned with two dimensional linear stable fields given by

$$u_{n,m} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h_{k,l} w_{n-k,m-l} \qquad (1)$$

where $(w_{n,m})_{(n,m)\in\mathbb{Z}^2}$ are i.i.d. S α S random variables with scale parameter $\sigma_w > 0$. As infinite summations are involved in the above expression, the existence of $u_{n,m}$ needs to be studied more precisely. It can be shown ([7]) that the series in (1) converges absolutely almost surely if and only if

$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |h_{k,l}|^{\alpha} < \infty, \quad \text{when} \quad 0 < \alpha < 1$$
$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |h_{k,l}| \left| \ln \min\left(|h_{k,l}|\sigma_w, \frac{1}{2}\right) \right| < \infty,$$
$$\text{when} \quad \alpha = 1$$
$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |h_{k,l}| < \infty, \quad \text{when} \quad \alpha > 1.$$

Furthermore, when $\alpha > 1$, a necessary and sufficient condition for the series defining $u_{n,m}$ to be convergent in the sense of the norm $\|\cdot\|_{\alpha}$ (or equivalently, in the L^p sense if $p < \alpha$) is:

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}|h_{k,l}|^{\alpha}<\infty.$$
 (2)

Note that this condition reduces to

$$(h_{k,l})_{(k,l)\in\mathbb{Z}^2}\in\ell^2(\mathbb{Z}^2)$$

when $(w_{n,m})_{(n,m)\in\mathbb{Z}^2}$ is a zero-mean white Gaussian noise.

When one of the above existence conditions is satisfied, the filtered random field $u_{n,m}$ is S α S with scale parameter

$$\sigma_u = \sigma_w \Big(\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |h_{k,l}|^{\alpha} \Big)^{1/\alpha}.$$

4. 2D FRACTIONAL α-STABLE PROCESSES

Our objective is now to define a 2D discrete-space process having long range dependence properties.

We recall that, in the 1D Gaussian case, such a process u_n is provided by an FARIMA(0, d, 0)with $d \in \mathbb{R}_+$ [1], which satisfies:

$$\left(1-q^{-1}\right)^d u_n = w_n$$

where w_n is a zero-mean white Gaussian noise with variance $2^{2d}\eta_w^2$ and $q^{\pm 1}u_n = u_{n\pm 1}$. As $(1 - q^{-1})$ is a discrete-time derivation operator, $(1 - q^{-1})^d$ represents a discrete-time fractional derivation of order $d(\in (0, 1/2))$. The power spectrum density of the process is given by

$$S(\omega) = \frac{\eta_w^2}{\left|\sin\frac{\omega}{2}\right|^{2d}}$$

and consequently, this random process can be viewed as the ouput of a linear filter with frequency response $|\sin(\omega/2)|^{-d}$ driven by a white Gaussian noise. Note that the long range dependence behaviour is caused by the divergence of the frequency response at $\omega = 0$.

By analogy, we define a 2D fractional stable process $u_{n,m}$ as the output of a bidimensional filter with frequency response

$$H_d(\omega_x, \omega_y) = \frac{1}{\left(\sin^2 \frac{\omega_x}{2} + \sin^2 \frac{\omega_y}{2}\right)^d} \quad (3)$$

whose input is an S α S iid sequence $w_{n,m}$ with scale parameter σ_w . When d < 1, H_d belongs to $L^1([-\pi,\pi]^2)$ and it can be proved that the impulse response $h_{k,l}$ of the filter is such that

$$h_{k,l} = O((k^2 + l^2)^{d-1})$$

when $k^2 + l^2 \rightarrow \infty$. We deduce that, for a > 0,

$$\sum_{(k,l)\in\mathbb{Z}^2}|h_{k,l}|^a<\infty$$

if d < 1 - 1/a. As the long dependence property is desired (d > 0), the conditions stated in the previous section show that the existence of $u_{n,m}$ is guaranteed only if $\alpha > 1$. In this case, the domain of validity of d is the interval $(0, 1 - 1/\alpha)$. Note that this result is consistent with both those existing for 1D stable FARIMA processes and those recently established for 2D Gaussian processes with long range dependence [3, 4].

5. ESTIMATION OF THE PARAMETERS

Let $u_{n,m}$ be an $N \times N$ fractional field as defined in the previous section and let $w_{n,m}$ be the associated driving noise. By computing the 2D DCT of $(u_{n,m})_{0 \le n,m \le N}$ and $(w_{n,m})_{0 \le n,m \le N}$, we obtain $S\alpha S$ fields $(U_{k,l}^N)_{0 \le k,l \le N}$ and $(W_{k,l}^N)_{0 \le k,l \le N}$. The DCT allows us to realize a frequency analysis of the filtering relation existing between $u_{n,m}$ and $w_{n,m}$. Its advantage over the discrete Fourier transform is to generate real transformed coefficients whose probability distribution are easier to characterize.

More precisely, $W_{k,l}^N$ is an S α S random variable with scale parameter

$$\sigma_{W_{k,l}^{N}} = \sigma_{w} \left(\sum_{n=0}^{N-1} |C(k,n)|^{\alpha} \right)^{1/\alpha} \times \left(\sum_{m=0}^{N-1} |C(l,m)|^{\alpha} \right)^{1/\alpha}$$
(4)

where

$$C(k,n) = \sqrt{\frac{2-\delta(k)}{N}} \cos\left(\frac{\pi k(n+1/2)}{N}\right).$$

Furthermore, by using the fact that H_d is real, it can be shown that

$$\frac{U_{k,l}^N}{W_{k,l}^N} \underset{N \to \infty}{\sim} H_d\left(\omega_{k,N}, \omega_{l,N}\right)$$

with $\omega_{k,N} = \pi k/N$. This yields:

$$\frac{\sigma_{U_{k,l}^N}}{\sigma_{W_{k,l}^N}} \underset{N \to \infty}{\sim} H_d\left(\omega_{k,N}, \omega_{l,N}\right).$$
(5)

Besides, as $U_{k,l}^N$ is a S α S random variable, $\log |U_{k,l}^N|$ is a second order random variable [8] with mean

$$\mathbb{E}\left\{\log|U_{k,l}^{N}|\right\} = C_{E}\left(\frac{1}{\alpha} - 1\right) + \log\sigma_{U_{k,l}^{N}} \quad (6)$$

where C_E is the Euler's constant. By combining relations (3), (4), (5) and (6), we obtain

$$E\left\{ \log |U_{k,l}^N| \right\} \underset{N \to \infty}{\sim} K_N(\alpha) + \log \sigma_w - d \log \left(\sin^2(\omega_{k,N}) + \sin^2(\omega_{l,N}) \right)$$

where $K_N(\alpha)$ is a constant depending on the stability index. The above relation suggests the use of a linear regression method for the estimation of $\log \sigma_w$ and d.

Interestingly, in the case $\alpha = 2$, this least square approach is close to the log-periodogram method [2, 4] often used to identify Gaussian fractional models.

Note that, in this approach, the value of α has been assumed to be known. When α is unknown, it can be estimated using classical methods. In particular, the sample characteristic function method [8] provides a simple estimation of α .

6. EXPERIMENTAL RESULTS

In Figs. 1 and 2-3, two 256×256 realizations of fractional stable fields are shown.

In order to evaluate the performances of the identification method proposed in the previous section, a Monte Carlo study has been realized for different parameter values of the synthesized fields (see Table 1). We observe the overall good performances for the estimation of α and d. We note however that the variance of estimation of $\log \sigma_w$ tends to increase as α decreases.

7. REFERENCES

- J.R.M. Hosking, *Fractional differenc*ing, Biometrika, Vol. 68, No. 1,1981, pp. 165-176
- [2] P.J. Brockwell, R. A. Davis, *Time Series: Theory and Methods*, New York: Springer-Verlag, 1995
- [3] B. Pesquet-Popescu, P. Larzabal, Analyse de textures à l'aide de modèles anisotropes à longue dépendance, Colloque GRETSI, Grenoble, France, 15-19 Sept. 1997, pp. 635-638
- [4] B. Pesquet-Popescu Modélisation bidimensionnelle de processus non stationnaires et application à l'étude du fond sous-marin, Doctorat en Sciences de l'Ecole Normale Supérieure de Cachan, 1998
- [5] W. Feller, An Introduction to Probability Theory and Its Applications, vol. II, New York: Wiley, 1971
- [6] I.S. Reed, P.C. Lee, T.K. Truong, Spectral Representation of Fractional Brownian Motion in n Dimensions and its Properties, IEEE Transactions on Information Theory, Vol.41, No.5, Sept.1995, pp. 1439-1451.
- [7] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, New York: Chapman and Hall, 1994
- [8] M. Shao, C.L. Nikias, "Signal Processing with Fractional Lower Order Moments: Stable Processes and Their Applications", *Proceedings of the IEEE*, Vol. 8, No. 7, July 1993, pp. 984-1009

Par.		α	$\log \sigma_w$	d
Values		2	0	0.4
Est.	mean	1.9946	-0.0161	0.3964
	std	0.0077	0.0219	0.0048
Values		1.8	0	0.4
Est.	mean	1.8026	0.0120	0.3970
	std	0.0517	0.2119	0.0120
Values		1.8	0	0.2
Est.	mean	1.7999	0.0097	0.1984
	std	0.0118	0.2862	0.0141
Values		1.7	0	0.4
Est.	mean	1.7063	0.0266	0.3964
	std	0.0701	0.2948	0.0145

Table 1: Performances of the proposed estimation method (500 realizations of 256×256 field).



Figure 2: A realization of the proposed fractional model ($\alpha = 1.8, d = 0.4$).



Figure 1: A realization of the proposed fractional model in the Gaussian case ($\alpha = 2, d = 0.4$).



Figure 3: Same image as in Fig. 2 shown in log-scales.