LEVY PROCESSES FOR IMAGE MODELING

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ABSTRACT

Nonhomogenous random fields are known to be well adapted to modeling a wide class of images. Their computational complexity generally causes their lack of appeal, we propose a more efficient model capable of capturing textures, shapes, as well as jumps typically encountered in infra-red images. The so-called Levy Random fields as we show, can indeed represent a very well adapted alternative for inference applications and the like.

1. INTRODUCTION

Research interest in determining good models for classes of processes is neither too old for novelty nor new as hot topic. Modeling is in fact a classical topic of research in light of its importance in a vast array of applications, and its ubiquitous usefulness.

Our main focus in this paper is to propose a new model for a class of images (infra-red images) whose importance would be too lengthy to describe here. Our interest in these models is for Bayesian inference and ultimately for object recognition. The challenge in image analysis is the wealth of features (edges, texture, shapes, etc.) which in turn is far too complicated to be captured by a simple statistical model. We show that the newly proposed model, namely the Levy model, can efficiently and effectively integrate these various statistics into one model.

Using the Levy-Khinchine formula [1, 2], every infinitely divisible distribution $\mu(dx)$ on $\mathbb{R}^2$ has a characteristic $\hat{\mu}(\phi) = \int_{\mathbb{R}^2} e^{i\phi(x)}\mu(dx)$ with the following general form, where $\phi(\cdot)$ is a bounded, compactly supported measurable function on $\mathbb{R}^2$,

$$\begin{aligned}\hat{\mu}(\phi) = exp\{ia(\phi) \Leftrightarrow \tfrac{\sigma^2(\phi)}{2} + \\ \int_{\mathbb{R}^2}(e^{iu\phi(s)} \Leftrightarrow 1)\nu(duds)\}\end{aligned} \tag{1}$$

where a is a continuous function and σ^2 is positive continuous function on $\mathbb{R}^2$, $\nu(duds)$ is the Levy measure on $\mathbb{R}\times\mathbb{R}^2$, subject to some technical condition. We can assume the first component $a(\phi) = 0$ since it is easy to treat. The above equation suggests [3] the representation of a Levy random field $X_t, t \in \mathbb{R}^2_+$ as a stochastic integral

$$X_t = \int\int_{\mathbb{R}\times[0,t]} uH(duds) + B_t$$

where B_t is a Gaussian random field corresponding to the second component of Equation 1. Given that a Gaussian random field is well known, we concentrate here on the first term of the above equation. The function $H(\cdot,\cdot)$ is a Poisson measure on $\mathbb{R}\times\mathbb{R}^2_+$ which has mean $E[(H(duds))] = \nu(duds)$. This measure here, controls the amplitudes as well as the rate of the jumps, and in fact defines a specific Levy Random field (e.g. as Poisson random field , Gamma random field and α -stable random field, etc.). When only positive jumps are present, the field is referred to as a subordinate random field, resulting in the above formula being rewritten as

$$X_t = \int\int_{\mathbb{R}_+\times[0,t]} uH_1(duds)$$

where $H_1(dsdu)$ ($H_2(dsdu)$ below) now controls only the positive (negative) jumps and their rate.

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This in turn, leads to an efficient simulation of a subordinate random field. Random fields with negative jumps can similarly be described as

$$X_t = \int\int_{\mathbb{R}_- \times [0,t]} u H_2(duds).$$

By introducing a probabilistic mixing of the two processes which we show to still satisfy the property of a Levy field, we are able to demonstrate as partially shown in the examples below, that realistic sample paths are obtained, much like those encountered in Infra-red imagery.

2. ILM FOR MEASURES ABSOLUTELY CONTINOUS WITH RESPECT TO OTHER LEVY MEASURES

2.1. Inverse Levy Measure

Our goal in this paper is to extend the ILM algorithm to afford us the modeling of a wider class of processes. Towards that end, we propose to consider a new measure $\mu(dx)$ resulting from a transformation of the original Levy measure $\nu(dx)$.

Specifically, suppose $\nu(dx)$ is a given Levy measure, then according to [3] for any real fixed $a \geq 0$, we are to find

$$u = L^{-1}(a) \stackrel{\text{def}}{=} \inf\{u' \geq 0 | L(u') \leq a\}, \qquad (2)$$

where we define

$$L(u) \stackrel{\text{def}}{=} \nu([u; +\infty)). \qquad (3)$$

The solution to (2) can be pretty difficult if at all tractable to obtain when the measure $\nu(dx)$ is of a general class. For clarity and tractability we restrict our study to the class of Levy measures for which these inversions are attainable.

As we show below, if $\nu(dx)$ is a Levy measure such that (2) can be solved, then

$$u = M^{-1}(a) \stackrel{\text{def}}{=} \inf\{u' \geq 0 | M(u') \leq a\} \qquad (4)$$

where $a \geq 0$ is arbitrary but fixed and

$$M(u) \stackrel{\text{def}}{=} \mu([u; +\infty)) \qquad (5)$$

can be solved so long as the measure $\mu(dx)$ is absolutely continuous with respect to $\nu(dx)$.

2.2. Generalized ILM

Let us suppose that $\nu(dx)$ and $\mu(dx)$ are two Levy measures. Let also $f(x)$ be a non-negative function defined on the non-negative half line, i.e. $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and

$$\mu(A) = \int_A f(x)\ \nu(dx)$$

for each set A measurable with respect to $\nu(dx)$ (i.e. the measure $\mu(dx)$ is absolutely continuous with respect to $\nu(dx)$). Since our interest is in sample paths on finite intervals ,we can then, with no loss of generality, assume that $f(x)$ is a finite staircase function. That means that

$$\exists\ n \in \mathbb{N}$$
$$\exists\ 0 = x_1 < \cdots < x_n < x_{n+1} = +\infty$$
$$\exists\ c_1, \ldots, c_n \in \mathbb{R}_+$$

$$\forall x \in \mathbb{R}_+\ f(x) = \textstyle\sum_{i=1}^n c_i \mathbf{I}_{\{[x_i, x_{i+1})\}}.$$

Proposition 1. *Given a fixed $a \geq 0$, the solution to*

$$u = \inf\{u' \geq 0 | M(u') \leq a\}.$$

is given by

$$u = L^{-1}\left(\frac{a \Leftrightarrow M(x_{i^a+1})}{c_{i^a}} + L(x_{i^a+1})\right),$$

where

$$i^a = \inf\{i \in \{1 \ldots n\}\ \ | a \in [M(x_{i+1}), M(x_i)\}.$$

Proof:

The above notation will become clear as we elaborate further. Denoting

$$a_i = M(x_i),$$

and in light of the non-negativity of the function $f(x)$ and of the measure $\nu(dx)$, $\{a_i\}_{i=1}^n$ is a non-increasing sequence. Using this fact we find the interval $(x_i, x_{i+1}]$ where our solution u lies and where i depends on a . Denote

$$i^a = \inf\{i \in \{1 \ldots n\}\ \ | a \in [a_{i+1}, a_i)\}, \qquad (6)$$

and note that although a_i decreases, it is still possible that $\forall i \in \{1 \dots n\}$ $a_i > a$ and hence there is no i such that $a \in [a_{i+1}, a_i)$. Consequently, that means that $u > x_n$ and thus $\forall x \in [u, +\infty)$ $f(x) = c_n \neq 0$. Thus we can write

$$\begin{aligned} u &= \inf\{u' | M(u') \leq a\} = \inf\{u' | c_n L(u') \leq a\} \\ &= \inf\{u' | L(u') \leq \frac{a}{c_n}\} = L^{-1}(\frac{a}{c_n}). \end{aligned}$$

Now let us suppose that $\{i \in \{1 \dots n\} \quad | \ a \in [a_{i+1}, a_i) \leq a\} \neq \emptyset$, which immediately yields $u \in (x_{i^a}, x_{i^a+1}]$. Indeed, we have chosen i^a so that $u \leq x_{i^a+1}$. At the same time if $u \leq x_{i^a}$, then due to the fact that $\mu(dx)$ is a non-negative measure we also have a contradiction with (6). Denoting

$$a' = a \Leftrightarrow M(x_{i^a+1}), \tag{7}$$

we obtain the following

$$u = L^{-1}(\frac{a'}{c_{i^a}} + L(x_{i^a+1})), \tag{8}$$

where the division is always possible per (6) where $c_{i^a} \neq 0$.

Remark: A few words for clarifying the last two steps are in order. Note that $\forall u \in (x_{i^a}, x_{i^a+1})$ $f(u) = c_{i^a} = \text{const}$ (Recall that we have already restricted the "domain" of u to the set $(x_{i^a}, x_{i^a+1}]$. The case $u = x_{i^a+1}$ will be considered later.) Then the following sequence of equivalent inequalities can be easily written:

$$\begin{aligned} (\mu([u', +\infty)) \leq a) \Leftrightarrow (\mu([u', +\infty)) \\ \Leftrightarrow \mu([x_{i^a+1}, +\infty)) \leq \\ a \Leftrightarrow \mu([x_{i^a+1}, +\infty))) \Leftrightarrow (\mu([u', x_{i^a+1})) \leq a \\ \Leftrightarrow M(x_{i^a+1})) \Leftrightarrow \\ (\mu([u', x_{i^a+1})) \leq a') \Leftrightarrow \\ (c_{i^a}\nu([u', x_{i^a+1})) \leq a') \Leftrightarrow \\ (\nu([u', x_{i^a+1})) \leq \frac{a'}{c_{i^a}}) \Leftrightarrow (\nu([u', x_{i^a+1}))+ \\ \nu([x_{i^a+1}, +\infty)) \leq \frac{a'}{c_{i^a}} + \nu([x_{i^a+1}, +\infty))) \Leftrightarrow \\ (\nu([u', +\infty)) \leq \frac{a'}{c_{i^a}} + L(x_{i^a+1})) \end{aligned}$$

to finally lead to

$$\begin{aligned} &\inf\{u' \mid \mu([u', +\infty)) \leq a\} = \\ &\inf\{u' \mid \nu([u', +\infty)) \leq \tfrac{a'}{c_{i^a}} + L(x_{i^a+1})\}. \end{aligned}$$

This clearly also leads to the solution of $L^{-1}(\frac{a'}{c_{i^a}} + L(x_{i^a+1}))$. Now recall that all these inequalities make sense if it is assumed that $u < x_{i^a+1}$. Nevertheless it is easy to see that the resulting formula also works in the case when $u = x_{i^a+1}$. We just get $u = L^{-1}(L(x_{i^a+1})$ which is true.

In summary we propose the following algorithm to compute $M^{-1}(a)$:

Algorithm.

1. For a given $a > 0$ and a staircase function $f(x)$ find the number i^a of the partition segment where the solution u lies using the formula (6);

2. The solution to the Equation (4) is then given by the formula (8) where a' is defined as in (7).

3. EXPERIMENTAL RESULTS

Finally we show some practical results that can be obtained using the method described above. Pictures on the left correspond to the initial Levy processes with some measures $\nu(dx)$. The ones on the right are related to the processes with the modified measures $\mu = \int f \, d\nu$ for some functions $f(x)$.

4. CONCLUSION

We proposed a Levy random field model capable of capturing and accounting for the various features typically encountered in Infra-red images (not limited to, however) which are of interest to us in a number of applications.

5. REFERENCES

[1] P. Levy, *Processus Stochastiques et Mouvement Brownien*. Editions J. Gabay, second ed., 1992.

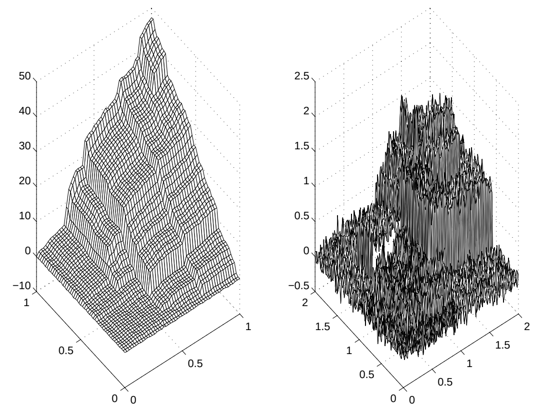

Figure 1: A three dimensional representation of the random field.

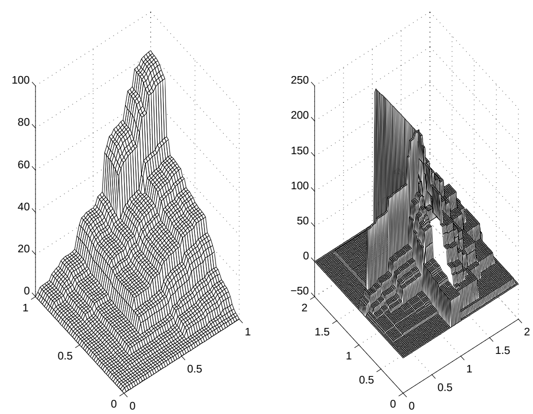

Figure 2: A three dimensional representation with localized compact features.

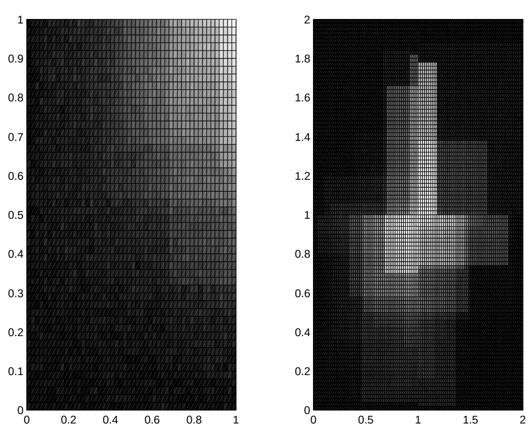

Figure 3: A 2-D random field with a given Levy measure and its mapping.

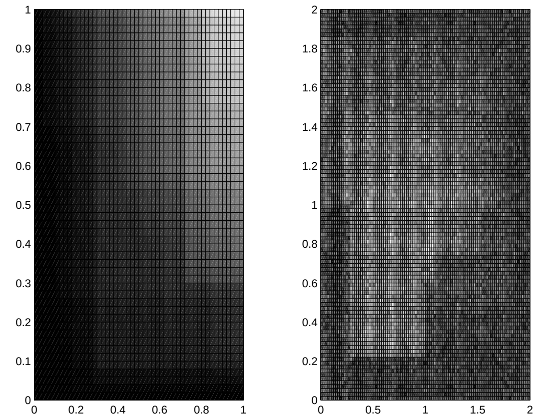

Figure 4: A 2-D random field with a given Levy measure and a different mapping functional.

[2] A. Janicki and A. Weron, *Simulation And Chaotic Behavior Of Alpha Stable Stochastic Processes*. 270 Madison Avenue, New York, New York 10016: Marcel Dekker, Inc., 1994.

[3] R. L. Wolpert and K. Ickstadt, "Simulation of levy random field," *Practical Nonparamatric and Semiparametric Bayesian Statistics*, vol. 133, pp. 227–242, January 1998.