ERROR SURFACES OF NORMALIZED BLIND CHANNEL EQUALIZATION ALGORITHMS

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ABSTRACT

Some recent *normalized* constant modulus algorithms possess surprisingly improved error performance surfaces (EPSs). In this paper, we look at the EPSs of these algorithms in the symbol-spaced setting. The lack of channel disparity in the fractionally-spaced setting is also considered. It is found that normalization has a decreasing effect as the equalizer length is increased and causes the stationary points to drift. While the drift is small for the Desirable Local Solutions (DLSs), it is more significant for the Undesirable Local Solutions (ULSs) some of which cease to exist.

1. INTRODUCTION

Experimental evaluation of many *fractionally-spaced* blind channel equalization algorithms conclude with problems of robustness when real world signals are used. For instance the condition that the subchannels must have no common zeros is frequently violated in mobile communication channels [1]. Since the channel disparity is lost, ULSs exist and we are ironically back to the basic limitation of *symbol-spaced* blind equalizers. At this point, one can safely anticipate that any algorithm with improved performance in the symbol-spaced setting will also perform better in the fractionally-spaced setting [2].

In this paper, the results in [3, 4] are extended. We use the following assumptions; **A1)** The channel input $a(k)$ is an i.i.d, stationary, real and binary signal, **A2)** There is no channel noise, **A3)** The channel and equalizer filters for the symbol spaced setting are respectively

$$H(q^{-1}) = \sum_{i=0}^{n} h_i q^{-i}, \;\; W(q^{-1}) = \sum_{j=0}^{m} w_j q^{-j} \qquad (1)$$

where h_i, w_j are real, $(m,n) < \infty$ and q^{-1} is the unit delay, **A4)** The channel does not have any transmission zeros.

The relationship between the equalizer input vector,

$$\mathbf{x}^{\top}(k) = [x(k)x(k \Leftrightarrow 1)...x(k \Leftrightarrow m+1)] \qquad (2)$$

and the transmitted symbols is $\mathbf{x}(k) = \mathbf{H}\mathbf{a}(k)$, where $\mathbf{H}$ is the $(m+1)\times(m+n+1)$ Sylvester Convolution (SC) matrix and

$$\mathbf{a}^{\top}(k) = [a(k)a(k \Leftrightarrow 1)...a(k \Leftrightarrow m \Leftrightarrow n+1)] \qquad (3)$$

The equalizer output is defined as $y(k) = \mathbf{x}^{\top}(k)\mathbf{w}(k)$, where $\mathbf{w}(k)$ is the equalizer coefficient vector.

The SCS-1 [3, 4], unnormalized SCS-1 (USCS-1) and CM algorithms are considered whose update equations are respectively given by

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \frac{\mu}{\|\mathbf{x}(k)\|_2^2}\left(1 \Leftrightarrow |y(k)|\right) y(k)\mathbf{x}(k) \qquad (4)$$

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu\left(1 \Leftrightarrow |y(k)|\right) y(k)\mathbf{x}(k) \qquad (5)$$

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \mu\left(1 \Leftrightarrow y^2(k)\right) y(k)\mathbf{x}(k) \qquad (6)$$

2. ERROR PERFORMANCE SURFACES

Remark 1 Under A1, A2 and A3, the distribution of the channel output, $x(k)$, is discrete and symmetrical. Furthermore, the joint distribution of the equalizer input vectors is described by

$$(\mathcal{X}, \mathcal{P}) \stackrel{\text{def}}{=} \left\{(\mathbf{x}_{[1]}, p_{[1]}), (\mathbf{x}_{[2]}, p_{[2]}), \cdots, (\mathbf{x}_{[D]}, p_{[D]})\right\} \qquad (7)$$

with the probabilities $p_{[m]} \stackrel{\text{def}}{=} P(\mathbf{x}(k) = \mathbf{x}_{[m]} \cup \mathbf{x}(k) = \Leftrightarrow \mathbf{x}_{[m]})$, since the gradient and the Hessian expressions to follow are not influenced by the sign of $\mathbf{x}(k)$. □

The mean gradient vector corresponding to (4) is given by

$$\mathbf{g}_S(\mathbf{w}) = E\left\{\left(|y(k)| \Leftrightarrow 1\right) y(k) \frac{\mathbf{x}(k)}{\|\mathbf{x}(k)\|_2^2}\right\} \qquad (8)$$

which by using Remark 1 can be converted into

$$\mathbf{g}_S(\mathbf{w}) = \sum_{m=1}^{D} p_{[m]}\left(|\mathbf{x}_{[m]}^{\top}\mathbf{w}| \Leftrightarrow 1\right)\frac{\mathbf{x}_{[m]}\mathbf{x}_{[m]}^{\top}}{\|\mathbf{x}_{[m]}\|_2^2}\mathbf{w} \qquad (9)$$

Any stationary point $\mathbf{w}_*$ of the SCS-1 algorithm satisfies $\mathbf{g}_S(\mathbf{w}_*) = \mathbf{0}$. Let us define

$$\mathbf{\Gamma}_S \stackrel{\text{def}}{=} \sum_{m=1}^{D} p_{[m]}\frac{\mathbf{x}_{[m]}\mathbf{x}_{[m]}^{\top}}{\|\mathbf{x}_{[m]}\|_2^2}, \mathbf{\Delta}_S(\mathbf{w}) \stackrel{\text{def}}{=} \sum_{m=1}^{D} p_{[m]}|\mathbf{x}_{[m]}^{\top}\mathbf{w}|\frac{\mathbf{x}_{[m]}\mathbf{x}_{[m]}^{\top}}{\|\mathbf{x}_{[m]}\|_2^2} \qquad (10)$$

then $\mathbf{g}_S(\mathbf{w}) = [\mathbf{\Delta}_S(\mathbf{w}) \Leftrightarrow \mathbf{\Gamma}_S]\mathbf{w}$, and the Hessian matrix is $\mathcal{H}_S(\mathbf{w}) \stackrel{\text{def}}{=} \frac{\partial \mathbf{g}_S^{\top}(\mathbf{w})}{\partial \mathbf{w}} = 2\mathbf{\Delta}_S(\mathbf{w}) \Leftrightarrow \mathbf{\Gamma}_S$. Similarly, we have for the USCS-1 and the CM algorithms; $\mathbf{g}_U(\mathbf{w}) = [\mathbf{\Delta}_U(\mathbf{w}) \Leftrightarrow \mathbf{\Gamma}]\mathbf{w}$, $\mathcal{H}_U(\mathbf{w}) = 2\mathbf{\Delta}_U(\mathbf{w}) \Leftrightarrow \mathbf{\Gamma}$, $\mathbf{g}_G(\mathbf{w}) = [\mathbf{\Delta}_G(\mathbf{w}) \Leftrightarrow \mathbf{\Gamma}]\mathbf{w}$, and $\mathcal{H}_G(\mathbf{w}) = 3\mathbf{\Delta}_G(\mathbf{w}) \Leftrightarrow \mathbf{\Gamma}$, where

$$\mathbf{\Gamma} \stackrel{\text{def}}{=} \sum_{m=1}^{D} p_{[m]}\mathbf{x}_{[m]}\mathbf{x}_{[m]}^{\top}, \mathbf{\Delta}_U(\mathbf{w}) \stackrel{\text{def}}{=} \sum_{m=1}^{D} p_{[m]}|\mathbf{x}_{[m]}^{\top}\mathbf{w}|\mathbf{x}_{[m]}\mathbf{x}_{[m]}^{\top} \qquad (11)$$

$$\mathbf{\Delta}_G(\mathbf{w}) \stackrel{\text{def}}{=} \sum_{m=1}^{D} p_{[m]}(\mathbf{x}_{[m]}^{\top}\mathbf{w})^2\mathbf{x}_{[m]}\mathbf{x}_{[m]}^{\top} \qquad (12)$$

The nature of each individual stationary point on the EPS is determined from the eigenvalues of the Hessian matrix as follows. Let $\Lambda \stackrel{\text{def}}{=} \{\lambda : \lambda = \text{eig}[\mathcal{H}(\mathbf{w}_*)]\}$ then for $(\lambda_i, \lambda_j) \in \Lambda$, we have the following cases; $\{\forall \lambda_i > 0\} \Leftrightarrow$

$\mathbf{w}_*$ local minimum, $\{\forall\lambda_i < 0\} \Leftrightarrow \mathbf{w}_*$ local maximum, $\{\exists(\lambda_i > 0, \lambda_j < 0)\} \Leftrightarrow \mathbf{w}_*$ unstable, $\mathcal{H}$ is *indefinite* and $\{\exists\lambda_i = 0\} \Leftrightarrow \mathbf{w}_*$ degenerate[1].

3. VANISHING LOCAL MINIMA OF NORMALIZED ALGORITHMS

In this section, we first demonstrate a special case where ULSs are avoided by normalizing the gradient vector. We then discuss the effect of normalization in the general case.

Case 1$(n = 1, m = 1)$ Take $h_0 = 1$. Then, $H(q^{-1}) = 1 + h_1 q^{-1}$, $W(q^{-1}) = w_0 + q^{-1} w_1$. Due to A4; $h_1 \neq \pm 1$, and therefore $D = 4$, $p_{[i]} = 1/4$. Let $\tilde{h}_1 \stackrel{\text{def}}{=} \frac{1-h_1}{1+h_1}$. Then,

$$\mathcal{X} = (1+h_1) \times \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\tilde{h}_1 \end{bmatrix}, \begin{bmatrix} \tilde{h}_1 \\ 1 \end{bmatrix}, \begin{bmatrix} \tilde{h}_1 \\ -\tilde{h}_1 \end{bmatrix} \right\} \tag{13}$$

It is easy to show that $\mathbf{\Gamma}_S = \frac{1}{2}\mathbf{I}$ and

$$\mathbf{\Gamma} = \begin{bmatrix} \frac{2(1+\tilde{h}_1^2)}{(1+\tilde{h}_1)^2} & \frac{1-\tilde{h}_1}{1+\tilde{h}_1} \\ \frac{1-\tilde{h}_1}{1+\tilde{h}_1} & \frac{2(1+\tilde{h}_1^2)}{(1+\tilde{h}_1)^2} \end{bmatrix} \tag{14}$$

Let $\mathbf{M}_i \stackrel{\text{def}}{=} \frac{1}{(1+h_1)^2}\mathbf{x}_{[i]}\mathbf{x}_{[i]}^\top$. We then have

$$\begin{aligned} \mathbf{\Delta}_S(\mathbf{w}) &= \tfrac{1}{2|1+\tilde{h}_1|}\Big(\tfrac{|w_0+w_1|}{2}\mathbf{M}_1 + \tfrac{|w_0-\tilde{h}_1 w_1|}{1+\tilde{h}_1^2}\mathbf{M}_2 \\ &+ \tfrac{|\tilde{h}_1 w_0 + w_1|}{1+\tilde{h}_1^2}\mathbf{M}_3 + \tfrac{|w_0-w_1|}{2|\tilde{h}_1|}\mathbf{M}_4\Big) \end{aligned} \tag{15}$$

$$\begin{aligned} \mathbf{\Delta}_U(\mathbf{w}) &= \tfrac{2}{|1+\tilde{h}_1|^3}\big(|w_0+w_1|\mathbf{M}_1 + |w_0-\tilde{h}_1 w_1|\mathbf{M}_2 \\ &+ |\tilde{h}_1 w_0+w_1|\mathbf{M}_3 + |\tilde{h}_1||w_0-w_1|\mathbf{M}_4\big) \end{aligned} \tag{16}$$

$$\begin{aligned} \mathbf{\Delta}_G(\mathbf{w}) &= \tfrac{4}{(1+\tilde{h}_1)^4}\big((w_0+w_1)^2\mathbf{M}_1 + (w_0-\tilde{h}_1 w_1)^2\mathbf{M}_2 \\ &+ (\tilde{h}_1 w_0+w_1)^2\mathbf{M}_3 + \tilde{h}_1^2(w_0-w_1)^2\mathbf{M}_4\big) \end{aligned} \tag{17}$$

Both components of $\mathbf{g}_{S,U,G}(\mathbf{w}) = \mathbf{0}$ are shown in Fig. 1 for $h_1 = \{0.4, 0.47, 0.6\}$. The shaded regions indicate *indefinite* Hessian matrices. The intersections of the solid and dashed lines outside the shaded regions denote the stable stationary points on the EPSs. The DLSs are marked as "D" and the ULSs are marked as "U". The DLSs lead to minimum InterSymbol Interference (ISI) after equalization. All other solutions are labeled as undesirable. All solutions in this special case lead to "open-eye" at the output, since $H(q^{-1}) = 1 + h_1 q^{-1}, h_1 \neq 1$. However, note how the ULSs vanish for the SCS-1 algorithm when h_1 is varied from 0.4 to 0.47. In fact, the two components that null the gradient vector only meet at two points for $h_1 > 0.45$. In the USCS-1 and CM algorithms, however, the ULSs exist regardless of the value of h_1.

Case 2$(1 < (n, m) < \infty)$ Our objective is to describe the underlying mechanism leading to the above results where some ULSs cease to exist for the SCS-1 algorithm. Here we take $\|\mathbf{h}\|_2^2 = \sum_{i=0}^n h_i^2 = 1$ without loss of generality. By using

$$x(k) = \sum_{j=0}^n h_j a(k-j) \tag{18}$$

the normalization term can be written as

$$\begin{aligned} \|\mathbf{x}(k)\|_2^2 &= \sum_{j=0}^n \sum_{l=0}^n h_j h_l \sum_{i=0}^m a(k-j-i)a(k-l-i) \\ &= (m+1)\,(1+c_m(k)) \end{aligned} \tag{19}$$

where

$$c_m(k) \stackrel{\text{def}}{=} \sum_{j=0}^n \sum_{l=0, l\neq j}^n h_j h_l \frac{\tilde{\mathbf{a}}^\top(k-j)\tilde{\mathbf{a}}(k-l)}{m+1} \tag{20}$$

and $\tilde{\mathbf{a}}(k) \stackrel{\text{def}}{=} [a(k)a(k-1)\cdots a(k-m+1)]^\top$.

Remark 2 For binary, i.i.d $a(k)$; $E\{c_m(k)\} = 0$ and
i) $c_m(k) > -1$ since $\|\mathbf{x}(k)\|_2^2 > 0$,
ii) $\lim_{m\to\infty} c_m(k) = \sum\sum h_j h_l \underbrace{E\{a(k-j)a(k-l)\}}_{=0, l\neq j} = 0$,
iii) $n = 1 \Rightarrow \forall m; c_m(k) < 1$,
iv) $n > 1$, for sufficiently large m, $P(c_m(k) > 1) \approx 0$. □

Remark 2.ii indicates that the variation of the normalization, $\|\mathbf{x}(k)\|_2^2$, will be less significant compared to its mean value for large m. Therefore, the normalization has a decreasing effect as the equalizer length is increased. Remark 2.iii can be shown by using standard vector inequalities. Showing Remark2.iv requires some sort of statistical model for the distribution at the channel output. Closed form distributions are not available but the conditional Gaussian model in [5], and the series expansion of the associated distribution in ([6], pp.93-95) can be used.

From (5), the stationary points of the USCS-1 algorithm satisfy

$$\mathbf{g}_U(\mathbf{w}_*) = E\{(|y(k)|-1)y(k)\mathbf{x}(k)\} = \mathbf{0} \tag{21}$$

By using $\mathbf{x}(k) = \mathbf{H}\mathbf{a}(k)$ and $y(k) = \mathbf{x}^\top(k)\mathbf{w}_*$ in the above

$$\mathbf{H}\underbrace{E\{(|y(k)|-1)\mathbf{a}(k)\mathbf{a}^\top(k)\}\mathbf{s}_*}_{\stackrel{\text{def}}{=}\mathbf{v}_U} = \mathbf{0} \tag{22}$$

where $\mathbf{s}_* \stackrel{\text{def}}{=} \mathbf{H}^\top\mathbf{w}_*$ is the parametrization in the combined space corresponding to $\mathbf{w}_*$. For the symbol-spaced setting no finite dimensional equalizer parametrization ensures $|y(k)| = 1$ and all stationary points are such that $\mathbf{v}_U \in \mathcal{N}(\mathbf{H})$ [2].

The stationary points of the SCS-1 algorithm satisfy

$$\mathbf{g}_S(\mathbf{w}_*) = E\left\{(|y(k)|-1)\,y(k)\frac{\mathbf{x}(k)}{\|\mathbf{x}(k)\|_2^2}\right\} = \mathbf{0} \tag{23}$$

Using $\mathbf{x}(k) = \mathbf{H}\mathbf{a}(k)$, $y(k) = \mathbf{x}^\top(k)\mathbf{w}_*$ and $\mathbf{s}_* = \mathbf{H}^\top\mathbf{w}_*$

$$\mathbf{H}E\left\{(|y(k)|-1)\frac{\mathbf{a}(k)\mathbf{a}^\top(k)}{\|\mathbf{x}(k)\|_2^2}\right\}\mathbf{s}_* = \mathbf{0} \tag{24}$$

Therefore, we are interested in $\mathbf{w}_*$ such that

$$\mathbf{v}_S \stackrel{\text{def}}{=} E\left\{(|y(k)|-1)\frac{\mathbf{a}(k)\mathbf{a}^\top(k)}{1+c_m(k)}\right\}\mathbf{s}_* \in \mathcal{N}(\mathbf{H}) \tag{25}$$

where we have dropped the constant $m+1$ in (19), since it has no influence on the nature of the EPS and can be regarded as a part of the step-size, μ, of the adaptation procedure. Taking m sufficiently large so that $|c_m(k)| < 1$ from

[1] A dense neighborhood of $\mathbf{w}_*$ satisfies $\mathrm{g}(\mathbf{w}_*) = \mathbf{0}$.

[2] $\mathcal{N}(\mathbf{H})$ denotes the null-space of $\mathbf{H}$.

Remark 2.i,2.iv, we use $\frac{1}{1+c_m(k)} = 1 + \sum_{i=1}^{\infty} i!(\Leftrightarrow 1)^i c_m^i(k)$ in (25) and get

$$\mathbf{v}_S = \mathbf{v}_U + \sum_{i=1}^{\infty} i!(\Leftrightarrow 1)^i \mathbf{v}_i \qquad (26)$$

where $\mathbf{v}_i \overset{\text{def}}{=} E\left\{c_m^i(k)(|y(k)| \Leftrightarrow 1)\mathbf{a}(k)\mathbf{a}^\top(k)\right\}\mathbf{s}_*, i = 1, \ldots$. For a stationary point, $\mathbf{w}_*$, of the USCS-1 algorithm, we have $\mathbf{H}\ \mathbf{v}_U = \mathbf{0}$. For the same $\mathbf{w}_*$, however, the normalized algorithm has

$$\mathbf{H}\ \mathbf{v}_S = \underbrace{\mathbf{H}\ \mathbf{v}_U}_{=\mathbf{0}} + \sum_{i=1}^{\infty} i!(\Leftrightarrow 1)^i\ \mathbf{H}\ \mathbf{v}_i \overset{?}{=} \mathbf{0} \qquad (27)$$

and in general the second equality in (27) is not satisfied. Therefore, the stationary points of the unnormalized algorithm drift when normalization is introduced. This can be verified by inspecting Fig. 1-(b),(c). The amount of drift varies according to the nature of the local solutions being desirable or undesirable. From the definition of $\mathbf{v}_i$, the drift of the DLSs, where $|y(k)|$ fluctuates closest to unity, will be less significant compared to the drift encountered by the ULSs where $|y(k)|$ is considerably less than unity. Some ULSs may disappear all together as in Fig. 1-(e),(f) or Fig. 1-(h),(i).

4. LACK OF CHANNEL DISPARITY AND NORMALIZATION IN THE FRACTIONALLY-SPACED SETTING

For simplicity, consider a fractionally-spaced setting where there are two subchannels $H_1(q^{-1})$, $H_2(q^{-1})$ with equal length and the equalizer filters $W_1(q^{-1})$, $W_2(q^{-1})$ satisfy the length condition. Assume that the subchannels fail the zero condition and the common zeros of $H_1(q^{-1})$, $H_2(q^{-1})$ are denoted by $T(q^{-1})$. In other words, $H_1(q^{-1}) = \tilde{H}_1(q^{-1})T(q^{-1})$, $H_2(q^{-1}) = \tilde{H}_2(q^{-1})T(q^{-1})$. Then, the gradient vectors corresponding to the unnormalized SCS-1 algorithm at a stationary point $\mathbf{W}_* = (\mathbf{w}_{1,*}, \mathbf{w}_{2,*})$ is

$$\mathbf{G}_U(\mathbf{W}_*) = \begin{bmatrix} \tilde{\mathbf{H}}_1 \\ \tilde{\mathbf{H}}_2 \end{bmatrix} \mathbf{T} E\{(|y(k)| \Leftrightarrow 1)\mathbf{a}(k)\mathbf{a}^\top(k)\}\mathbf{s}_* = \mathbf{0} \qquad (28)$$

where $\tilde{\mathbf{H}}_1$, $\tilde{\mathbf{H}}_2$, and $\mathbf{T}$ are SC matrices and

$$y(k) = \mathbf{x}_1^\top(k)\mathbf{w}_{1,*} + \mathbf{x}_2^\top(k)\mathbf{w}_{2,*} \qquad (29)$$

$$\mathbf{s}_* = \mathbf{T}^\top(\tilde{\mathbf{H}}_1^\top \mathbf{w}_{1,*} + \tilde{\mathbf{H}}_2^\top \mathbf{w}_{2,*}) \qquad (30)$$

Since $\tilde{H}_1(q^{-1})$ and $\tilde{H}_2(q^{-1})$ do not have any common zeros, it is sufficient to look at

$$\breve{\mathbf{G}}_U(\mathbf{W}_*) = \mathbf{T} E\{(|y(k)| \Leftrightarrow 1)\mathbf{a}(k)\mathbf{a}^\top(k)\}\mathbf{s}_* = \mathbf{0} \qquad (31)$$

Similarly, for the normalized algorithm

$$\breve{\mathbf{G}}_S(\mathbf{W}_*) = \mathbf{T} E\left\{\frac{(|y(k)| \Leftrightarrow 1)\mathbf{a}(k)\mathbf{a}^\top(k)}{\mathbf{a}^\top(k)(\mathbf{H}_1^\top\mathbf{H}_1 + \mathbf{H}_2^\top\mathbf{H}_2)\mathbf{a}(k)}\right\}\mathbf{s}_* = \mathbf{0} \qquad (32)$$

If the length of the equalizers is sufficiently long, we can expand the denominator in (32) and reach the same observation that a drift will be introduced to all stationary points if normalization is in place and the drift is most significant for the worst undesirable stationary point of the unnormalized algorithm. Therefore, normalized algorithms are expected to have better convergence properties in the fractionally-spaced setting. This confirms the results in [2], where the CM algorithm fails to open the eye under lack of channel disparity, but the normalized CM or the SCS-1 algorithms open the eye after a small number of iterations.

5. CONCLUSIONS

This paper shows that normalization of the gradient vector of a gradient-descent adaptive algorithm (as in NLMS) has significant merits on multimodal EPSs encountered in blind equalization. It is demonstrated that ULSs may cease to exist due to normalization. The results suggest that normalization will increase the robustness of blind fractionally-spaced equalizers under lack of channel disparity confirming earlier experimental results [2].

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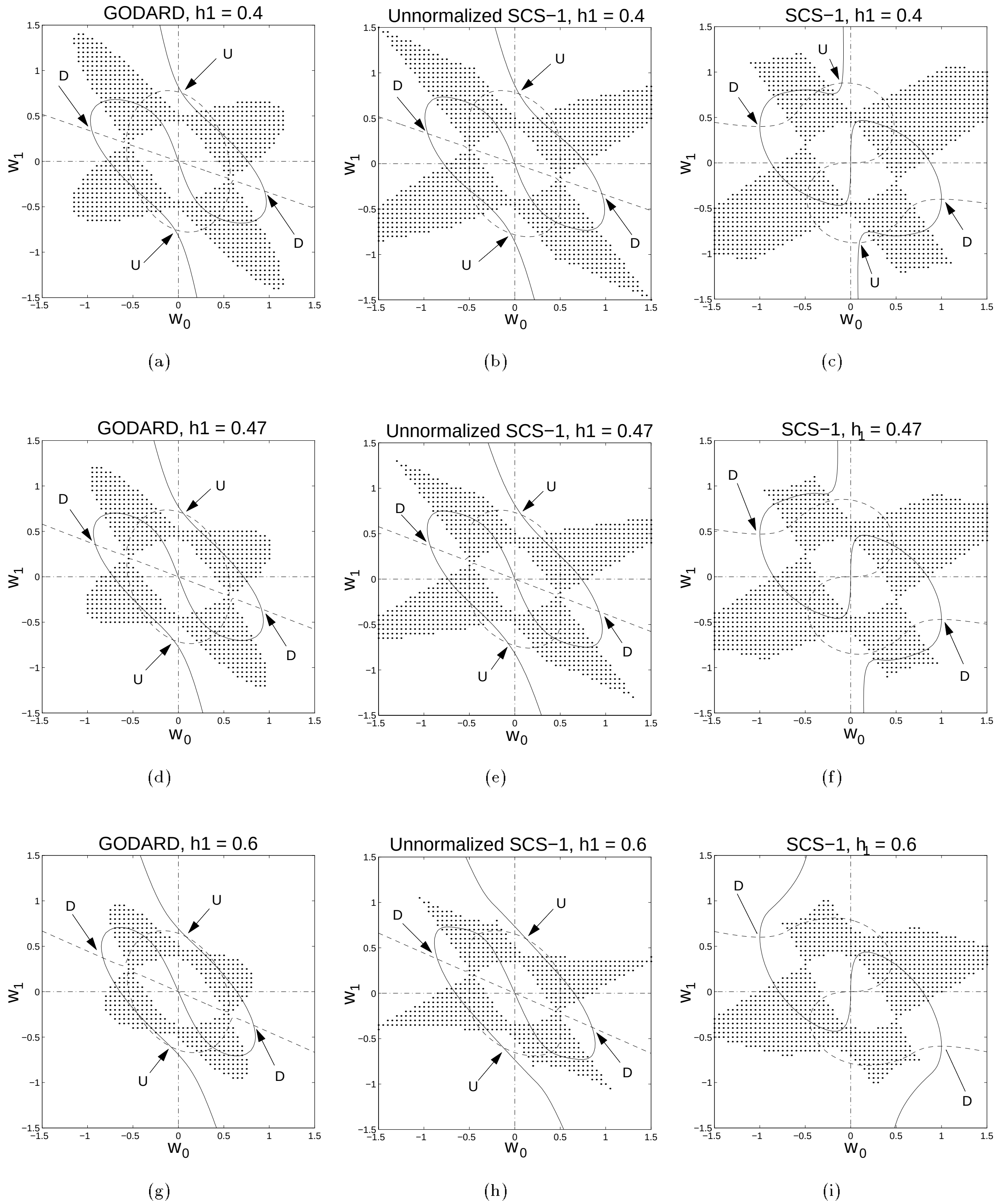

Figure 1. Minima of the Godard (CM), Unnormalized SCS-1 and SCS-1 algorithms for a two-tap channel: ($h_1 = 0.3$:(a),(b),(c)), ($h_1 = 0.47$:(d),(e),(f)), ($h_1 = 0.6$:(g),(h),(i)), and a two-tap equalizer. “D”: desirable, “U”: undesirable solutions.