ODIF FOR MORPHOLOGICAL FILTERS

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ABSTRACT

In this paper our recently introduced method called output distributional influence function (ODIF) is used for the evaluation of the robustness properties of the morphological filters. Several examples of the ODIFs for the dilation, the closing, and the clos-opening are given and explained carefully. For each of these morphological filters the effect of filter length is examined by using the ODIFs for the expectation and the variance. The robustness properties of the filters are also compared to each other and the effect of the distribution of the contamination is investigated for the closing as an example of realistic filtering conditions.

1. INFLUENCE FUNCTION

Influence function (IF) is a useful heuristic tool of robust statistics introduced by Hampel [1, 2] under the name influence curve (IC) for studying the performance of filters under noisy conditions.

Definition 1. The IF of estimator T at underlying probability distribution F is given by

$$\mathrm{IF}(x;T,F) = \lim_{t\to 0^+} \frac{T\left((1-t)F + t\Delta_x\right) - T(F)}{t}$$

for those x where this limit exists.

In this definition Δ_x is the probability measure which puts mass 1 at the point x. The IF gives the effect that an infinitesimal contamination at point x has on the estimator T when divided by the mass of the contamination. So the IF gives asymptotic bias caused by the contamination and thus characterizes properties of the estimator as the number of observations approaches infinity.

2. CHANGE-OF-VARIANCE FUNCTION

The IF gives only one aspect of robustness of an estimator, namely local robustness of the asymptotic value of the estimator. Another important aspect is the local robustness of the asymptotic variance. Asymptotic variance of estimator T at F denoted by $V(T,F)$ is defined to be the variance of $\sqrt{N}\,[T(F_N) - T(F)]$ as $N \to \infty$, where F_N is the empirical distribution of sample $(X_1, X_2, \ldots, X_N)$. Local robustness of the asymptotic variance can be characterized by the change-of-variance function (CVF) defined as follows, [3].

Definition 2. The CVF of estimator T at F is defined as

$$\mathrm{CVF}(x;T,F) = \lim_{t\to 0^+} \frac{V(T,(1-t)F + t\Delta_x) - V(T,F)}{t}$$

for those x where this limit exists.

3. OUTPUT DISTRIBUTIONAL INFLUENCE FUNCTION

Since the IF is an asymptotic measure, it describes properties of infinite length filters which may differ from those of finite length filters used in the real world filtering applications. It would be more useful and more interesting to examine properties of these finite length filters rather than the asymptotic properties. In the case where the output distribution of a filter can be expressed in a closed form as a function of the distribution functions of the input samples we introduced in [5] output distributional influence function (ODIF) for analysing the robustness of the finite length filters. It is also required for the IF and the CVF that the filters are Fisher consistent, i.e., they measure asymptotically the correct value. This prohibits the analysis of morphological filters by these asymptotic methods but the ODIFs do not have this limitation.

We assume here that the input samples are independent and identically distributed (i.i.d.) random variables. First we need a way to denote the output distribution function of a filter when a fraction ε of the input samples has different distribution than the rest of the samples. We denote by $H_{(1-\varepsilon)F+\varepsilon G_y}(\cdot)$ the output distribution $H_F(\cdot)$ of the filter where every occurrence of the common distribution function F of the input samples is replaced by $(1-\varepsilon)F+\varepsilon G_y$ and G_y can be any distribution function with mean y. As usual, we define $h_{(1-\varepsilon)F+\varepsilon G_y}(x) = \frac{d}{dx}H_{(1-\varepsilon)F+\varepsilon G_y}(x)$. We gave the following definition for the ODIF for the distribution function in [5].

Definition 3. Let the output distribution function of a filter be $H_F(\cdot)$ where $F(\cdot)$ is the common distribution function of the input samples and let $G_y(\cdot)$ be a distribution function having mean y. Then the ODIF for the distribution function $\Omega(\cdot)$ is

$$\Omega(x,y) = \lim_{\varepsilon\to 0^+} \frac{H_{(1-\varepsilon)F+\varepsilon G_y}(x) - H_F(x)}{\varepsilon}$$

for those x and y where this limit exists.

If the output distribution function $H_F(\cdot)$ does not contain any derivative of F, the ODIF for the distribution function $\Omega(\cdot)$ can be expressed as, [5]

$$\Omega(x,y) = \frac{h_F(x)}{f(x)}\left(G_y(x) - F(x)\right). \tag{1}$$

In [5] we defined the ODIF in the same way as for the distribution function in Definition 3 also for the density function and moments. The ODIFs for the density function, the expectation, and the variance were derived to be

$$\omega(x,y) = \frac{d}{dx}\Omega(x,y), \quad (2)$$

$$\omega_\mu(y) = \int_{-\infty}^{\infty} x\omega(x,y)dx, \quad (3)$$

and

$$\omega_{\sigma^2}(y) = \int_{-\infty}^{\infty} x^2\omega(x,y)dx - 2\mu_{H_F}\omega_\mu(y), \quad (4)$$

where μ_{H_F} is the mean of the distribution H_F.

4. OUTPUT DISTRIBUTIONS OF ONE-DIMENSIONAL MORPHOLOGICAL FILTERS

Analytical expressions for the output distribution functions of the dilation, the closing, and the clos-opening are given in [4] for a discrete signal s whose values are i.i.d. random variables having common distribution function $F(\cdot)$ and for a convex one-dimensional structuring set B of length N. The distribution functions for the dilation, the closing, and the clos-opening are

$$\begin{aligned} H_{F,\mathrm{dil}}(x) &= F(x)^N, \\ H_{F,\mathrm{clo}}(x) &= NF(x)^N - (N-1)F(x)^{N+1}, \end{aligned}$$

and, when $N > 2$,

$$\begin{aligned} H_{F,\mathrm{co}}(x) = {} & \frac{N^2-N-2}{2}F(x)^{2N+2} \\ & +(-N^2+N+1)F(x)^{2N+1} \\ & +\frac{N^2-N}{2}F(x)^{2N} - (N-1)F(x)^{N+1} \\ & +NF(x)^N. \end{aligned}$$

The density functions obtained by differentiation are for the dilation, the closing, and the clos-opening

$$\begin{aligned} h_{F,\mathrm{dil}}(x) &= NF(x)^{N-1}f(x), \\ h_{F,\mathrm{clo}}(x) &= \left[N^2 - (N^2-1)F(x)\right]F(x)^{N-1}f(x), \end{aligned}$$

and, when $N > 2$,

$$\begin{aligned} h_{F,\mathrm{co}}(x) = {} & \big[(N^2-N-2)(N+1)F(x)^{N+2} \\ & +(-N^2+N+1)(2N+1)F(x)^{N+1} \\ & +(N-1)N^2F(x)^N - (N^2-1)F(x) \\ & +N^2\big]F(x)^{N-1}f(x). \end{aligned}$$

5. ODIFS FOR DILATION

The ODIFs for the distribution and the density functions of the dilation $\Omega_{\mathrm{dil}}(\cdot)$ and $\omega_{\mathrm{dil}}(\cdot)$ for a convex one-dimensional structuring set B of length N are by equations (1) and (2)

$$\Omega_{\mathrm{dil}}(x,y) = NF(x)^{N-1}\left(G_y(x) - F(x)\right)$$

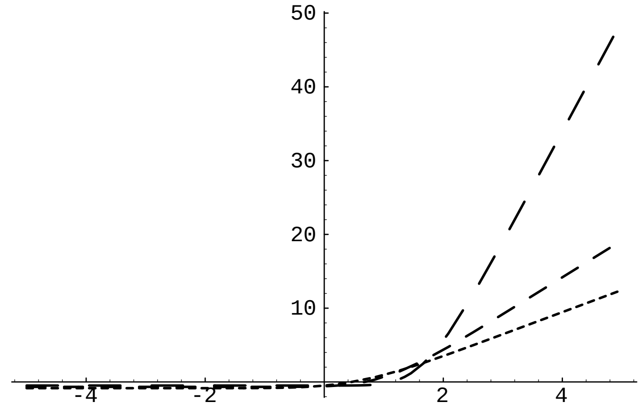

Figure 1: The ODIFs for the expectation of the dilation for a convex one-dimensional structuring set B of lengths 3 (short dashes), 5 (medium dashes), and 15 (long dashes) at $F = \Phi$ and $G_y = \Delta_y$.

and

$$\begin{aligned} \omega_{\mathrm{dil}}(x,y) = {} & N(N-1)F(x)^{N-2}f(x)\left(G_y(x) - F(x)\right) \\ & +NF(x)^{N-1}\left(g_y(x) - f(x)\right). \end{aligned}$$

Now by equation (3) we obtain the ODIF for the expectation of the dilation $\omega_{\mu,\mathrm{dil}}(\cdot)$ for a convex one-dimensional structuring set B of length N and $G_y = \Delta_y$ as

$$\begin{aligned} \omega_{\mu,\mathrm{dil}}(y) = {} & N(N-1)\int_y^{\infty} xF(x)^{N-2}f(x)dx \\ & -N^2\int_{-\infty}^{\infty} xF(x)^{N-1}f(x)dx + NyF(y)^{N-1}. \end{aligned}$$

We denote by Φ and ϕ the distribution and the density functions of the standard normal distribution. The graphs of the above function $\omega_{\mu,\mathrm{dil}}(y)$ when $F = \Phi$ are shown in Figure 1 for three different lengths of the structuring set B. As can be seen from the left half of the Figure 1, negative outliers have fixed influence on the expectation of the filter. So any negative outlier, irrespective of its value, decreases the expectation of the output of the filter the same small fixed amount. This decrease is slightly smaller for the longer structuring set length N. The robustness against positive outliers is clearly worse. For the dilation the supremum of the absolute value, i.e., the worst influence which a small amount of contamination of fixed size can have on the value of the estimator, equals infinity. As the structuring set length increases the robustness against positive outliers gets worse as can be observed from Figure 1.

From equation (4) we obtain that the ODIF for the variance of the dilation $\omega_{\sigma^2,\mathrm{dil}}(\cdot)$ for a convex one-dimensional structuring set B of length N and $G_y = \Delta_y$ is

$$\begin{aligned} \omega_{\sigma^2,\mathrm{dil}}(y) = {} & N(N-1)\int_y^{\infty} x^2F(x)^{N-2}f(x)dx \\ & -N^2\int_{-\infty}^{\infty} x^2F(x)^{N-1}f(x)dx \\ & +Ny^2F(y)^{N-1} - 2\mu_{H_{F,\mathrm{dil}}}\omega_{\mu,\mathrm{dil}}(y). \end{aligned}$$

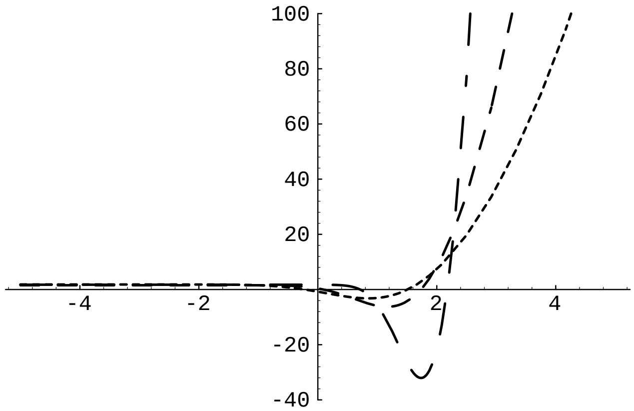

Figure 2: The ODIFs for the variance of the dilation multiplied by N for a convex one-dimensional structuring set B of lengths 3 (short dashes), 5 (medium dashes), and 15 (long dashes) at $F = \Phi$ and $G_y = \Delta_y$.

The graphs of $N\omega_{\sigma^2,\mathrm{dil}}(y)$ are plotted to Figure 2 for $F = \Phi$ and the same lengths of the structuring set B as in Figure 1. The minima points of the graphs can be found to be at $y = \mu_{H_{F,\mathrm{dil}}}$ by differentiating the above function $\omega_{\sigma^2,\mathrm{dil}}(y)$ with respect to y and finding the zero of the derivative. So the variance reduces when the contamination is near the expectation of the filter having no contamination. For smaller values of y the effect of the contamination is constant and very small but for larger values the graphs are not bounded above and the dilation is thus not robust against positive outliers either in the variance sense. Altogether the dilation has very poor robustness against positive outliers and should not be used when such outliers are possible.

6. ODIFS FOR CLOSING

The closing is the dilation followed by the erosion. The ODIFs for the distribution and the density functions of the closing $\Omega_{\mathrm{clo}}(\cdot)$ and $\omega_{\mathrm{clo}}(\cdot)$ for a convex one-dimensional structuring set B of length N are by equations (1) and (2)

$$\begin{aligned}&\Omega_{\mathrm{clo}}(x,y)\\ &\quad = \left[N^2-(N^2-1)F(x)\right]F(x)^{N-1}\left(G_y(x)-F(x)\right)\end{aligned}$$

and

$$\begin{aligned}\omega_{\mathrm{clo}}(x,y) \quad = \quad & N(N-1)\left[N-(N+1)F(x)\right]\\ & \times F(x)^{N-2}f(x)\left(G_y(x)-F(x)\right)\\ & +\left[N^2-(N^2-1)F(x)\right]F(x)^{N-1}\\ & \times\left(g_y(x)-f(x)\right).\end{aligned}$$

For the closing the ODIF for the expectation $\omega_{\mu,\mathrm{clo}}(\cdot)$ for a convex one-dimensional structuring set B of length N and $G_y = \Delta_y$ is by equation (3)

$$\begin{aligned}&\omega_{\mu,\mathrm{clo}}(y)\\ &= N(N-1)\int_y^\infty x\left[N-(N+1)F(x)\right]F(x)^{N-2}f(x)dx\\ &\quad -\int_{-\infty}^\infty x\left[N^3-(N+1)(N^2-1)F(x)\right]F(x)^{N-1}f(x)dx\\ &\quad +y\left[N^2-(N^2-1)F(y)\right]F(y)^{N-1}.\end{aligned}$$

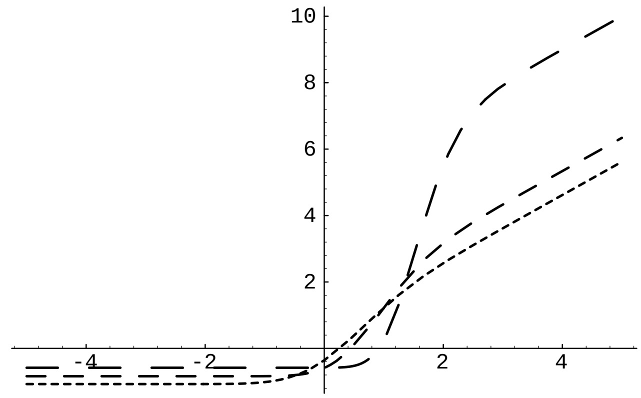

Figure 3: The ODIFs for the expectation of the closing for a convex one-dimensional structuring set B of lengths 3 (short dashes), 5 (medium dashes), and 15 (long dashes) at $F = \Phi$ and $G_y = \Delta_y$.

When $F = \Phi$, the ODIFs for the expectation of the closing are shown in Figure 3 for the same three lengths as in Figure 1 for the dilation. Again the negative outliers have small fixed influence which decreases when the length N of the structuring set increases and the positive outliers can also for the closing take the expectation over all limits. When the ODIFs for the expectation of the dilation and the closing are compared on the positive side, it can be noticed that after a short transition period the slopes of the curves become constant and the value of the slope for the dilation is N times the slope for the closing. So the influence of positive outliers is larger on the expectation of the dilation than on the expectation of the closing but the both expectations approach infinity when y approaches infinity.

From equation (4) we obtain that the ODIF for the variance of the closing $\omega_{\sigma^2,\mathrm{clo}}(\cdot)$ for a convex one-dimensional structuring set B of length N and $G_y = \Delta_y$ is

$$\begin{aligned}&\omega_{\sigma^2,\mathrm{clo}}(y)\\ &\quad = \quad N(N-1)\int_y^\infty x^2\left[N-(N+1)F(x)\right]F(x)^{N-2}f(x)dx\\ &\qquad -\int_{-\infty}^\infty x^2\left[N^3-(N+1)(N^2-1)F(x)\right]F(x)^{N-1}\\ &\qquad \times f(x)dx+y^2\left[N^2-(N^2-1)F(y)\right]F(y)^{N-1}\\ &\qquad -2\mu_{H_{F,\mathrm{clo}}}\omega_{\mu,\mathrm{clo}}(y).\end{aligned}$$

The graphs of $N\omega_{\sigma^2,\mathrm{clo}}(y)$ are shown in Figure 4 for $F = \Phi$ and three different lengths of the structuring set B. Similarly as for the dilation the minimum of each of the graphs is found to be at $y = \mu_{H_{F,\mathrm{clo}}}$. The graphs are quite similar to those of the dilation in Figure 1 otherwise but for the closing the growth of the ODIF for the variance multiplied by N on the positive side is slower than for the dilation.

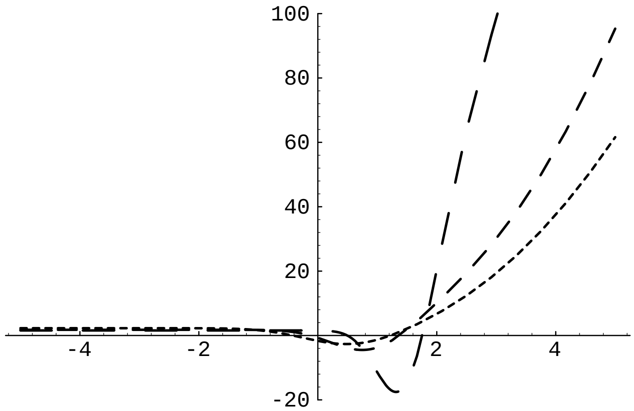

Figure 4: The ODIFs for the variance of the closing multiplied by N for a convex one-dimensional structuring set B of lengths 3 (short dashes), 5 (medium dashes), and 15 (long dashes) at $F = \Phi$ and $G_y = \Delta_y$.

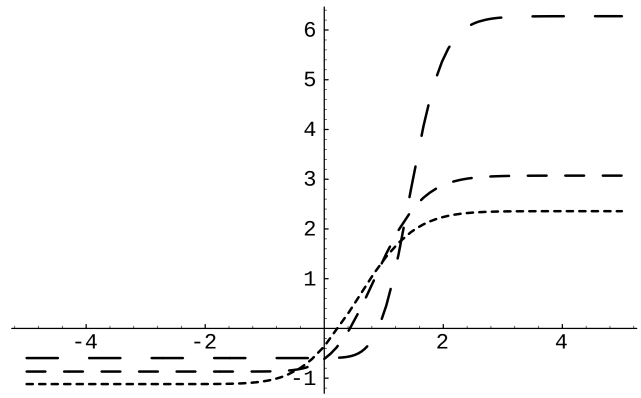

Figure 5: The ODIFs for the expectation of the clos-opening for a convex one-dimensional structuring set B of lengths 3 (short dashes), 5 (medium dashes), and 15 (long dashes) at $F = \Phi$ and $G_y = \Delta_y$.

7. ODIFS FOR CLOS-OPENING

The clos-opening is the closing followed by the opening and the ODIFs for the distribution and the density functions $\Omega_{\mathrm{co}}(\cdot)$ and $\omega_{\mathrm{co}}(\cdot)$ for a convex one-dimensional structuring set B of length $N > 2$ are by equations (1) and (2)

$$\begin{aligned} \Omega_{\mathrm{co}}(x,y) \quad = \quad & \big[(N^2-N-2)(N+1)F(x)^{N+2} \\ & +(-N^2+N+1)(2N+1)F(x)^{N+1} \\ & +(N-1)N^2F(x)^N-(N^2-1)F(x) \\ & +N^2\big]F(x)^{N-1}\left(G_y(x)-F(x)\right) \end{aligned}$$

and

$$\begin{aligned} & \omega_{\mathrm{co}}(x,y) \\ & \quad = \quad \big[(N^2-N-2)(N+1)(2N+1)F(x)^{N+2} \\ & \qquad +(-N^2+N+1)(2N+1)2NF(x)^{N+1} \\ & \qquad +(N-1)N^2(2N-1)F(x)^N-(N^2-1)NF(x) \\ & \qquad +N^2(N-1)\big]F(x)^{N-2}f(x)\left(G_y(x)-F(x)\right) \\ & \qquad +\big[(N^2-N-2)(N+1)F(x)^{N+2} \\ & \qquad +(-N^2+N+1)(2N+1)F(x)^{N+1} \\ & \qquad +(N-1)N^2F(x)^N-(N^2-1)F(x)+N^2\big] \\ & \qquad \times F(x)^{N-1}\left(g_y(x)-f(x)\right). \end{aligned}$$

Now the ODIF for the expectation of the clos-opening $\omega_{\mu,\mathrm{co}}(\cdot)$ for a convex one-dimensional structuring set B of length $N > 2$ is by equation (3)

$$\omega_{\mu,\mathrm{co}}(y) = \int_{-\infty}^{\infty} x\omega_{\mathrm{co}}(x,y)dx.$$

In Figure 5 are shown the graphs of the above function $\omega_{\mu,\mathrm{co}}(y)$ for three different values of N when $F = \Phi$ and $G_y = \Delta_y$. Behaviour of the graphs on the negative side is the similar as for the dilation and the closing. The most significant thing is that the clos-opening has limited supremum of the absolute value and thus the outliers have limited influence on the expectation. However, the supremum of the absolute value of the clos-opening increases as the length of the structuring set increases and approaches infinity when N approaches infinity. Clearly the clos-opening has better robustness properties than the dilation and the closing since the graphs are bounded also above.

From equation (4) we obtain that the ODIF for the variance of the clos-opening $\omega_{\sigma^2,\mathrm{co}}(\cdot)$ for a convex one-dimensional structuring set B of length N is

$$\omega_{\sigma^2,\mathrm{co}}(y) = \int_{-\infty}^{\infty} x^2\omega_{\mathrm{co}}(x,y)dx - 2\mu_{H_{F,\mathrm{co}}}\omega_{\mu,\mathrm{co}}(y).$$

The graphs in Figure 6 show $N\omega_{\sigma^2,\mathrm{co}}(y)$ for $F = \Phi$, $G_y = \Delta_y$, and three different lengths of the structuring set. Similarly as for the dilation and the closing the minima of the graphs are at $y = \mu_{H_{F,\mathrm{co}}}$ and the negative contamination has only small influence on the ODIF for the variance. The graphs of $N\omega_{\sigma^2,\mathrm{co}}(y)$ are bounded above which makes the clos-opening most robust of the three morphological filters considered thus far also in this sense.

8. EFFECT OF THE DISTRIBUTION G_y

In this section we consider instead of the delta distribution more realistic noise distributions G_y. We have selected the closing with the structuring set length 5 to illustrate the effect of the distribution G_y for the morphological filters.

We experimented the ODIFs for the expectation and the variance with several different distributions G_y. For the same variance of the contamination the results were very similar to each other. Due to lack of space we present here the results only for the normal distribution. In Figure 7 are shown the ODIFs for the expectation when $F = \Phi$, $N = 5$, and G_y is the delta distribution and the normal distribution with the variances $\frac{1}{2}$, 1, and 2. From this figure it can be seen that the graph with the smallest variance of the contamination is closest to the graph for the delta distribution and the graph with the largest variance furthest. In the area where the density function ϕ is practically zero the graphs join together but otherwise they differ from each other. The fact that the normally

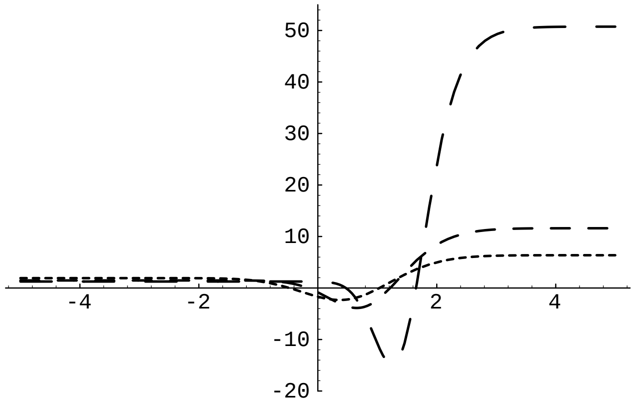

Figure 6: The ODIFs for the variance of the clos-opening multiplied by N for a convex one-dimensional structuring set B of lengths 3 (short dashes), 5 (medium dashes), and 15 (long dashes) at $F = \Phi$ and $G_y = \Delta_y$.

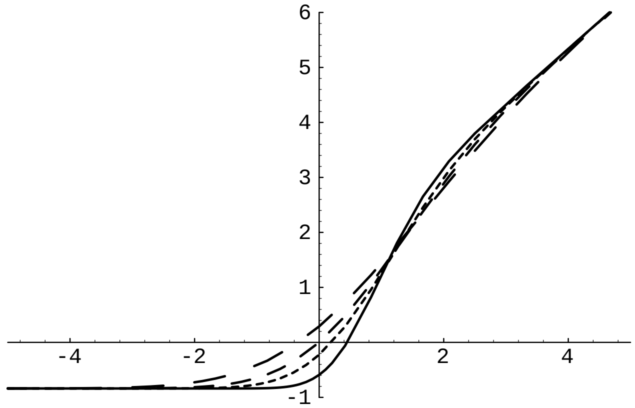

Figure 7: The ODIFs for the expectation of the closing for a convex one-dimensional structuring set B of length 5 at $F = \Phi$ when G_y is normal distribution with variance $\frac{1}{2}$ (short dashes), 1 (medium dashes), and 2 (long dashes) and when $G_y = \Delta_y$ (solid line).

distributed contaminations G_y increase also the probability of values near y causes different behaviour of the graphs. The higher the variance of the normally distributed contamination the wider the area where the probability increases. For smaller values of y this means that the ODIFs for expectation for the normally distributed contaminations are larger than for the delta distribution and larger for larger variance of contamination. For somewhat larger y the opposite behaviour is observed.

The ODIFs for the variance multiplied by N for $F = \Phi$, $N = 5$, and the same contaminations G_y as in Figure 7 are shown in Figure 8. This figure shows how for smaller variance of the normally distributed contamination G_y also the ODIF for the variance multiplied by N is closer to the graph for the delta distribution. When the variance of the normally distributed contaminaton G_y approaches zero the ODIF for the variance multiplied by N approaches the graph for the delta function. When the variance of the contamination is smaller than the variance of F, the ODIF for the variance multiplied by N has a negative part. Otherwise

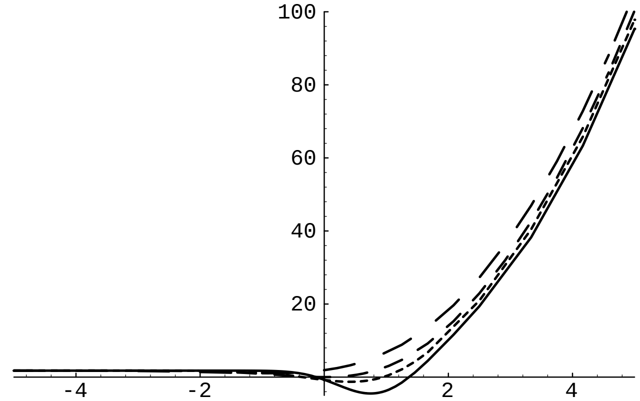

Figure 8: The ODIFs for the variance of the closing multiplied by N for a convex one-dimensional structuring set B of length 5 at $F = \Phi$ when G_y is normal distribution with variance $\frac{1}{2}$ (short dashes), 1 (medium dashes), and 2 (long dashes) and when $G_y = \Delta_y$ (solid line).

contamination never decreases the variance when compared to the variance of the filter having no contamination.

9. CONCLUSIONS

A recently introduced tool for determining the robustness of the finite length filters was used for the morphological filters. The ODIFs for the expectation and the ODIFs for the variance multiplied by N of the dilation, the closing, and the clos-opening were plotted for different structuring set lengths. These plots showed that the robustness of each of these filters against positive outliers is worse for longer structuring set length. When compared to each other the robustness of the dilation was found to be the worst and the robustness of the clos-opening the best. This confirms further the previous knowledge of these filters. The effect of the distribution of the contamination was also considered.

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