MORPHOLOGICAL SCALE-SPACE THEORY FOR SEGMENTATION PROBLEMS

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ABSTRACT

This work presents some new results on the morphological scale-space theory and their use in image segmentation. Basically, we introduce an idempotent smoothing operation based on the recently proposed multiscale-morphological-dilation-erosion method, and analyse some of its features concerned mainly with monotonicity and the way the image extrema merge in a multi-scale simplification process.

1. INTRODUCTION

The representation of an image by multiple scales has proved to be useful in a large number of image processing applications. New and interesting multiscale methods have been considered for extracting features of a signal. Recently, Jackway [1] proposed a morphological-based scale-space method that guarantees the monotone property for the extrema of an image. This property, inherent to the scale-space theory, means that the number of the signal features (the extrema set) decreases monotonicly as a function of scale. Thus, if a signal feature is present at a certain level of representation, then it can also be found in its finer representations, up to the original image (zero scale).

The morphological scale-space is based on the well-known non-linear morphological operations [3], and takes into account both positive and negative scales σ. For positive scales, the image is smoothed by dilation, and for negative ones it is processed by erosion. The magnitude of the parameter $|\sigma|$ represents the intuitive notion of scale. Let f be an image function defined in the discrete domain, $f : \mathcal{D}_f \subseteq Z^2 \rightarrow R$. A smoothed version of this image at scale σ is given by [1]

$$(f \otimes g_\sigma)(x) = \begin{cases} (f \oplus g_\sigma)(x) & \text{if } \sigma > 0; \\ f(x) & \text{if } \sigma = 0; \\ (f \ominus g_\sigma)(x) & \text{if } \sigma < 0; \end{cases} \qquad (1)$$

where $\oplus$ and $\ominus$ stand for grayscale dilation and erosion, respectively, and g_σ is a scaled structuring function, $g_\sigma : \mathcal{G}_\sigma \subseteq Z^2 \rightarrow R$. One can show that in order to verify the monotonic property, g_σ should be a non-positive, anticonvex, and even function with $g(0) = 0$ [1].

Theorem 1 *[2] Let the set of points $E_{max}(f) = \{ x \in f : x$ is a local maximum$\}$ and $E_{min}(f) = \{ x \in f : x$ is a local minimum$\}$ represent the extrema of image f. Then, for any scales $\sigma_2 < \sigma_1 < \sigma < \sigma_3 < \sigma_4$,*

$$E_{min}(f \otimes g_{\sigma_2}) \subseteq E_{min}(f \otimes g_{\sigma_1}) \subseteq E_{min}(f) \quad \textit{and}$$

$$E_{max}(f \otimes g_{\sigma_4}) \subseteq E_{max}(f \otimes g_{\sigma_3}) \subseteq E_{max}(f)$$

In his work, Jackway illustrates the use of the morphological scale-space method for reducing monotonicly the number of extrema (regional maxima or minima) of an image [1]. He also defines the watershed of a signal [3] smoothed at scale σ as the feature of interest. Nevertheless, as stated by the author, the method cannot be directly applied to image segmentation since "the watershed arcs move spatially with varying scale and are not a subset of those at zero scale" [1].

This work addresses this problem by analysing the way image extrema merge, throughout the different levels of representation, in order to obtain interesting segmentation results from the morphological scale-space approach.

Section 2 shows briefly how we use the set of markers defined at a certain scale to obtain an initial partition of the image, and discusses some aspects concerning the way image extrema merge across scales. Some properties related to the definition of a basic configuration of the original image are discussed in Section 3.

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Our conclusions are described in Section 4. Finally, appendices A and B present the proof of the propositions introduced in this paper.

2. MORPHOLOGICAL SCALE-SPACE AND SEGMENTATION

Images 1(a)-1(c) illustrate the algorithm proposed in [1] which defines a multiscale watershed set. Here, the structuring function is the circular paraboloid $g(x,y) = -(x^2+y^2)$. The algorithm first smoothes the original image f, obtaining $(f \otimes g_\sigma)$ (Eq. 1). Next, since we are considering a negative scale, we define the regional minima of $f \otimes g_\sigma$ as the set of markers to be used in a dual reconstruction (homotopy modification [3]) of the gradient image $|\bigtriangledown(f \otimes g_\sigma)|$. Finally, we compute the watershed lines of this modified image. As we can see in Fig. 1(c), these watershed lines can be useful for the image analysis, but do not delineate regions according to a common segmentation model.

The scale-space properties are explored here for defining a significant set of image extrema representing the markers for segmentation. Before we focus on this point, we need to consider the problem of "forcing" the watershed lines to follow the contour of the regions being segmented. This can simply be done by a dual reconstruction of the original image, taking as markers the set of minima obtained from the filtering at scale σ. Fig. 1(d) has the same set of regional minima as in Fig. 1(c). In our case, we use this set to reconstruct the original image, Fig. 1(a), and obtain the watershed of its gradient representing a better partition of this image (Fig. 1(d)).

The next section discusses some aspects concerning the way the extrema merge across the scale-space smoothing steps.

2.1. The minima/maxima minimal configuration set

Once we define the smoothed image, it is very difficult to characterize the set of extrema that remains (or should remain) at a certain scale. The following idempotence considerations constitute an important simplification of this set.

Let f be an image function as before and $(f \ominus g_\sigma)^n = \underbrace{(((f \ominus g_\sigma) \ominus g_\sigma) \ominus \cdots \ominus g_\sigma)}_{n\ times}$. We define an idempotent smoothed version of f, at scale σ, as

$$(f \odot g_\sigma)(x) = \begin{cases} (f \oplus g_\sigma)^n(x) & \text{if } \sigma > 0; \\ f(x) & \text{if } \sigma = 0; \\ (f \ominus g_\sigma)^n(x) & \text{if } \sigma < 0; \end{cases} \qquad (2)$$

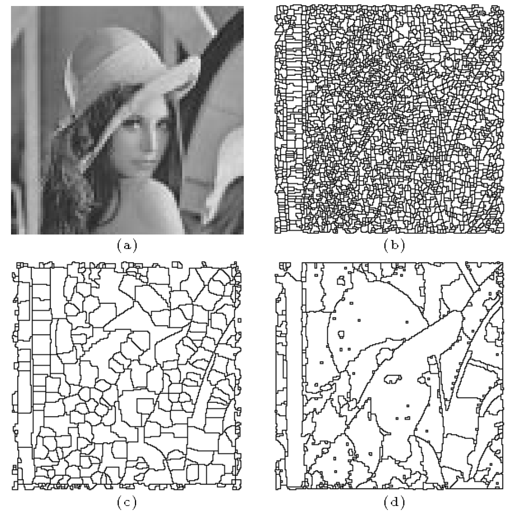

Figure 1: (a) Original image, and (b) its watershed lines. (c) The scale-space result for σ =-5, and (d) the scale-space with reconstruction of the original image.

where n is the number of iterations so that $(f \oplus g_\sigma)^n(x) = (f \oplus g_\sigma)^{n-1}(x)$ for $\sigma > 0$, and $(f \ominus g_\sigma)^n(x) = (f \ominus g_\sigma)^{n-1}(x)$ for $\sigma < 0$. The following two propositions concern the idempotence property of Eq. 2 (here, we consider only smoothing through the negative scales, the extension to the positive ones is obtained from duality). The proof of all the propositions discussed in this paper is given in appendices A and B.

Proposition 1 *For any $\sigma < 0$ there exists a value n such that $(f \ominus g_\sigma)^n(x) = (f \ominus g_\sigma)^{n-1}(x)$.*

Proposition 2 *For any scales $\sigma_2 < \sigma_1 < 0$, let m and n be the number of iterations such that $(f \ominus g_{\sigma_2})^m(x) = (f \ominus g_{\sigma_2})^{m-1}(x)$ and $(f \ominus g_{\sigma_1})^n(x) = (f \ominus g_{\sigma_1})^{n-1}(x)$. In this case, we have that $m \geq n$.*

The set of regional minima obtained after smoothing the image till idempotence constitutes the *minima minimal configuration - MMC* set at scale σ.

Let $(f \ominus g_\sigma)^n(x)$ define the MMC set at scale σ. The next proposition specifies the way two minima merge, during the smoothing operation, till we reach the MMC set.

Proposition 3 *Let x_i and $x_j \in E_{min}(f)$ denote two points of the image f with $f(x_i) < f(x_j)$. For a 4-connectivity and $\sigma < 0$, we can show that pixel x_j will belong to the influence zone ?? of x_i, $\mathcal{Z}(x_i)$, if $\exists x_k \in \mathcal{Z}(x_i)$ so that*

$$f(x_j) - f(x_k) \geq D_\sigma \times (d(x_j, x_k) - 1) \qquad (3)$$

where d denotes the city-block distance and $D_\sigma = |sup_{t \in \mathcal{G}_\sigma}(g_\sigma(t))|$, $t \neq 0$.

Shortly, the MMC set represents a simplification of the minima defined by the original morphological scale-space method (Eq. 1). This set, with less non-significant minima at a certain scale, can be used as a marker in a segmentation process. Observe that merging is a function of the distance between minima as well as their gray-scale value, and that it can be directly controlled by the structuring function g_σ. Fig. 2 illustrates such a segmentation based on the same number of regional minima used as markers. Fig. 3 shows another segmentation example.

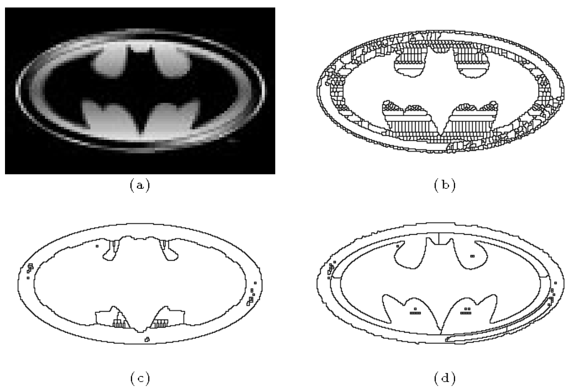

Figure 2: (a) Original image, and (b) its watershed lines. (c) The space-scale for σ =-8, and (d) the segmentation result.

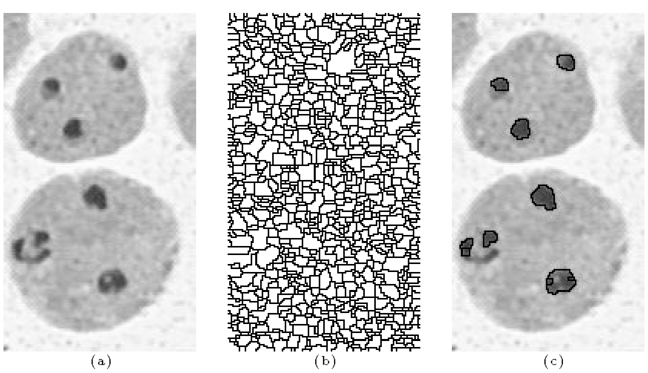

Figure 3: (a) Original image, and (b) its watershed lines. (c) The segmentation result for $\sigma = -1$.

Finally, we can also prove the following statement regarding computation time.

Proposition 4 *For discrete images, the MMC set can be obtained from Eq. 2 by considering a small* 3×3 *structuring function* g_σ.

3. PROPERTIES OF THE MMC SET

In this section, we discuss some basic properties of the morphological scale-space method, showing how they also hold for Eq. 2. The next result concerns the anti-extensivity of Eq. 2 for negative scales (a dual result for positive scales can be obtained from duality).

Proposition 5 *The definition of the MMC set is represented by the following properties:*

1. *For* $\sigma \to 0$, $(f \odot g_\sigma)(x) \to f(x)$ *for all* $x \in \mathcal{D}_f$.
2. *For* $\sigma \to -\infty$, $(f \odot g_\sigma)(x) \to \inf_{t \in \mathcal{D}_f}\{f(t)\}$ *for all* $x \in \mathcal{D}_f$.
3. *For* $\sigma_2 < \sigma_1 < 0$, $(f \odot g_{\sigma_2})(x) \leq (f \odot g_{\sigma_1})(x) \leq f$ *for all* $x \in \mathcal{D}_f$.

The next two propositions relate the value and the position of the minima in both smoothed and original images, across the different levels of representation.

Proposition 6 *Let the structuring function have a single maximum at the origin, that is,* $g_\sigma(x)$ *is a local maximum so it implies that* $x = 0$ *and then:*

If $\sigma < 0$ *and* $(f \odot g_\sigma)(x_{min})$ *is a local minimum, then* $f(x_{min})$ *is a local minimum and* $(f \odot g_\sigma)(x_{min}) = f(x_{min})$.

Proposition 7 *Let the structuring function have a single maximum at the origin, that is,* $g_\sigma(x)$ *is a local maximum so it implies that* $x = 0$ *and then:*

If $\sigma_2 < \sigma_1 < 0$ *and* $(f \odot g_{\sigma_1})(x_{min})$ *is a local minimum, then* $(f \odot g_{\sigma_2})(x_{min})$ *is a local minimum and* $(f \odot g_{\sigma_1})(x_{min}) = (f \odot g_{\sigma_2})(x_{min})$.

Based on the above considerations, we can also guarantee the monotonic property of the image extrema during the MMC set definition.

Proposition 8 *Let* $E_{min}(f) = \{x \in f : x \text{ is a regional minimum}\}$. *Then, we can prove that for any scales* $\sigma_2 < \sigma_1 < 0$,

$$E_{min}(f \odot g_{\sigma_2}) \subseteq E_{min}(f \odot g_{\sigma_1}) \subseteq E_{min}(f)$$

4. CONCLUSIONS

The work reported here considers the problem of using the morphological scale-space method for image segmentation. Our approach is based on the analysis and simplification of the extrema of an image, whose smoothed version is characterized by a monotonicly filtering of these extrema. Basically, we have defined an idempotent operation which allows an interesting representation of the images we can obtain at different scales. As illustrated here, this aspect, associated with the morphological reconstruction operation, can be considered to define sound segmentation algorithms based on the scale-space approach.

A. PROOF OF PROPOSITIONS

Proposition 1
Proof: From proposition 9 in Appendix B, we have that for a city-block distance, d, and $D_\sigma = |sup_{t\in\mathcal{G}_\sigma}(g_\sigma(t))|$, $t \neq 0$, any number of iterations $i \leq n$ of Eq. 2 is such that,

$$(f \ominus g_\sigma)^i(y) \leq f(x) + D_\sigma \times d(y,x) \tag{4}$$

if $d(y,x) \leq i$.

For any x and $y \in \mathcal{D}_f$, let $s = \sup(d(x,y))$. Thus, at iteration s, we have that

$$(f \ominus g_\sigma)^s(x) = \inf\{f(y) + D_\sigma \times d(y,x)\}. \tag{5}$$

Therefore, for any iteration $t > s$, $(f \ominus g_\sigma)^s(x) = (f \ominus g_\sigma)^t(x)$. Thus, we can say that $\exists n$ so that $(f \ominus g_\sigma)^n(x) = (f \ominus g_\sigma)^{n-1}(x)$, for any $x \in \mathcal{D}_f$.

□

Proposition 2
Proof: When $\sigma \to -\infty$ and $g_\sigma \to 0$, we have that $(f \odot g_\sigma)(x) = \inf_{t\in\mathcal{D}_f}\{f(t)\}$ for all $x \in \mathcal{D}_f$. In this case, the value of the global minimum of the image is propagated all over the image points and $n \leq \sup\{d(x,y)\}$ for any x and $y \in \mathcal{D}_f$.

When $\sigma \to 0$ and $g_\sigma \to \infty$, then $(f \odot g_\sigma)(x) = f(x)$ for all $x \in \mathcal{D}_f$. In this case, no minimum value is propagated on the image and $n = 1$. Thus, we have that the value of a point can be further propagated at scale σ_2 than σ_1, which yields $m \geq n$ for the scale order in proposition 2.

□

Proposition 3
Proof: If the point $x_j \notin E_{min}(f \odot g_\sigma)$, then $\exists y \in N_4(x_j, 3\times 3)$ [1] such that

$$(f \odot g_\sigma)(y) < (f \odot g_\sigma)(x_j) \leq f(x_j) \tag{6}$$

Since we consider that x_j will belong to the influence zone of x_i, $\mathcal{Z}(x_i)$, then

$$(f \odot g_\sigma)(y) = f(x_k) + D_\sigma \times d(y, x_k), \tag{7}$$

for any $x_k \in \mathcal{Z}(x_i)$. Thus, from Eq. 6 and 7 we have that merging will occur when

$$f(x_k) + D_\sigma \times d(y,x_k) \leq f(x_j) \tag{8}$$

Since $d(y,x_k) = d(x_j,x_k) - 1$, we have that

$$f(x_j) - f(x_k) \geq D_\sigma \times (d(x_j,x_k) - 1). \tag{9}$$

[1] $N_G(x,\alpha)$ is the set of G-connected points in the neighborhood α of x. $G = *$ represents the connectivity defined for the image extrema (4- or 8- connectivity).

□

Proposition 4
Proof: Let us assume two points x and y with city-block distance $d(x,y) = i$, $i > 1$. According to proposition 9, we have that at iteration i, $\exists z \in N_4(y, 3\times 3)$ with $d(z,x) = i-1$ such that

$$(f \ominus g_\sigma)^{i-1}(z) \leq f(x) + D_\sigma \times (i-1) \tag{10}$$

Therefore,

$$(f \ominus g_\sigma)^i(y) \leq (f \ominus g_\sigma)^{i-1}(z) + D_\sigma. \tag{11}$$

From Eq. 10 and 11 we have that

$$(f \ominus g_\sigma)^i(y) \leq f(x) + D_\sigma \times i \leq f(x) - g_\sigma(y-x), \tag{12}$$

since $-g_\sigma(y-x) \geq D_\sigma \times i$.

Therefore, if at any iteration $j < n$,

$$(f \ominus g_\sigma)^j(y) = f(x) - g_\sigma(y-x), \tag{13}$$

and knowing that at iteration n,

$$(f \ominus g_\sigma)^n(y) \leq f(x) + D_\sigma \times n \leq (f \ominus g_\sigma)^j(y), \tag{14}$$

thereby the computation of $inf_{t\in\mathcal{G}_\sigma}\{f(y-t) - g_\sigma(t)\}$ should not be considered for any x with $d(y,x) > 1$. In this case, we only need to take into account a 4-connected 3×3 structuring function, g_σ, in Eq. 2.

□

Proposition 5
Proof: From proposition 9 in Appendix B we have that for any $x \in \mathcal{D}_f$,

$$(f \ominus g_\sigma)^n(x) = \inf_{y\in\mathcal{D}_f}\{f(y) + D_\sigma \times d(x,y)\}, \text{ for any } y \in \mathcal{D}_f. \tag{15}$$

Based on this result, we can state the following

1. For $\sigma \to 0$, $D_\sigma \to \infty$, and the inf value occurs for y such that $d(x,y) = 0$, i.e., $x = y$. Thus, $(f \odot g_\sigma)(x) = f(x)$.
2. For $\sigma \to -\infty$, $D_\sigma \to 0$, and the inf value occurs for y corresponding to the global minimum of the image.
3. Since erosion is anti-extensive, $(f \ominus g_\sigma)^j \leq (f \ominus g_\sigma)^i$, for any $j \geq i$. Thus, according to proposition 10 in Appendix B,

$$(f \ominus g_{\sigma_2})^j(x) \leq (f \ominus g_{\sigma_2})^i(x) \leq (f \ominus g_{\sigma_1})^i(x), \tag{16}$$

and from propositon 2,

$$(f \odot g_{\sigma_2})(x) \leq (f \odot g_{\sigma_1})(x) \text{ for all } x \in D_f. \tag{17}$$

□

Proposition 6
Proof: *Base:* Theorem 1 holds for $n = 1$.

Step: Now, for $n > 1$, since $(f \ominus g_\sigma)^n(x) = ((f \ominus g_\sigma)^{n-1} \ominus g_\sigma)(x)$, from theorem 1 we also have that if $x_{min} \in E_{min}((f \ominus g_\sigma)^n)$, then $x_{min} \in E_{min}((f \ominus g_\sigma)^{n-1})$ and

$$(f \ominus g_\sigma)^n(x_{min}) = (f \ominus g_\sigma)^{n-1}(x_{min}) \qquad (18)$$

Since by hypothesis $x_{min} \in E_{min}((f \ominus g_\sigma)^{n-1})$ implies that $x_{min} \in E_{min}(f)$ and

$$(f \ominus g_\sigma)^{n-1}(x_{min}) = f(x_{min}), \qquad (19)$$

thus, from equations 18 and 19 and for $\sigma < 0$, we have that if $x_{min} \in E_{min}((f \ominus g_\sigma)^n)$, then $x_{min} \in E_{min}(f)$ and

$$(f \ominus g_\sigma)^n = (f \ominus g_\sigma)^{n-1} = f(x_{min}). \qquad (20)$$

□

Proposition 7
Proof: From proposition 9 in Appendix B, we have that for any $y \in \mathcal{D}_f$ and σ_2,

$$(f \odot g_\sigma)(y) = \inf_{x \in \mathcal{D}_f} \{f(x) + D_{\sigma_2} \times d(y,x)\}. \qquad (21)$$

Now, if $y \in E_{min}(f \odot g_{\sigma_2})$, the *inf* value occurs for $x = y$ and $f(y) < (f(x) + D_{\sigma_2} \times d(y,x))$ for any $x \in \mathcal{D}_f$. Since $D_{\sigma_2} < D_{\sigma_1}$, then $f(y) < (f(x) + D_{\sigma_1} \times d(y,x))$ and $y \in E_{min}(f \odot g_{\sigma_1})$.

□

Proposition 8
Proof: For any scales $\sigma_2 < \sigma_1 < 0$, let us suppose proposition 8 is false and $E_{min}(f \odot g_{\sigma_2}) \nsubseteq E_{min}(f \odot g_{\sigma_1})$. Then, there might be a point x_{min} in the image such that $x_{min} \in E_{min}(f \odot g_{\sigma_2})$ and $x_{min} \notin E_{min}(f \odot g_{\sigma_1})$, which contradicts proposition 7.

□

B. BASIC PROPOSITIONS

Proposition 9 *For any iteration $i \leq n$ of Eq. 2 and $\sigma < 0$, we have that*

$$(f \odot g_\sigma)^i(y) \leq f(x) + D_\sigma \times d(y,x) \qquad (22)$$

if $d(y,x) \leq i$, with d being the city-block distance and $D_\sigma = |sup_{t \in \mathcal{G}_\sigma}(g_\sigma(t))|$, $t \neq 0$.

Proof: *Base:* Since the *sup* value is given by the 4-connected points, $t \in \mathcal{G}_\sigma$, closer to the origin of the structuring function, then for $d(y,x) = 1$,

$$(f \ominus g_\sigma)^1(y) \leq f(x) - g_\sigma(x-y) = f(x) + D_\sigma. \qquad (23)$$

Step: If a point y is such that $d(y,x) = i$, then $\exists z \in N_4(y, 3 \times 3)$ with $d(z,x) = i - 1$ so that

$$\begin{aligned}(f \ominus g_\sigma)^i(y) &= inf_t\{(f \ominus g_\sigma)^{i-1}(y-t) - g_\sigma(t)\} \\ &\leq (f \ominus g_\sigma)^{i-1}(z) + D_\sigma. \qquad (24)\end{aligned}$$

Since by hypothesis we have the following

$$(f \ominus g_\sigma)^{i-1}(z) \leq f(x) + D_\sigma \times (i-1), \qquad (25)$$

then by replacing Eq. 25 in Eq. 24

$$(f \ominus g_\sigma)^i(y) \leq f(x) + D_\sigma \times i. \qquad (26)$$

□

Proposition 10 *For any scales $\sigma_2 < \sigma_1 < 0$ and any number of iterations i*

$$(f \ominus g_{\sigma_2})^i(x) \leq (f \ominus g_{\sigma_1})^i(x). \qquad (27)$$

Proof: *Base:* Since the scale-space erosion is anti-extensive, the inequality holds for $i = 1$ [2].

Step: By hypothesis we have that $(f \ominus g_{\sigma_2})^{i-1}(x) \leq (f \ominus g_{\sigma_1})^{i-1}(x)$. Since the scale-space erosion is an anti-extensive and decreasing operation [2], and

$$\begin{aligned}((f \ominus g_{\sigma_2})^{i-1} \ominus g_{\sigma_2})(x) &\leq ((f \ominus g_{\sigma_2})^{i-1} \ominus g_{\sigma_1})(x) \\ &\leq ((f \ominus g_{\sigma_1})^{i-1} \ominus g_{\sigma_1})(x), \qquad (28)\end{aligned}$$

then

$$(f \ominus g_{\sigma_2})^i(x) \leq (f \ominus g_{\sigma_1})^i(x). \qquad (29)$$

□

C. REFERENCES

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