FAST ALGORITHMS FOR FRACTIONAL FOURIER TRANSFORMS

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ABSTRACT

The *fractional Fourier transform* (FRFT) is a one-parametric generalization of the classical Fourier transform. The FRFT was introduced in the 80th and has found a lot of applications and is now used widely in signal processing. Both the space and the spatial frequency domains, respectively, are special cases of the fractional Fourier domains. They correspond to the 0th and 1st fractional Fourier domains, respectively. In this paper, we briefly introduce the multi-parametrical FRFT and its fast algorithms.

1. INTRODUCTION

Fourier analysis is one of the most frequently used tools in signal processing and in many other scientific disciplines. In the mathematical literature a generalization of the Fourier transform known as the *fractional Fourier transform* $\mathcal{F}^\alpha$ (FRFT) was proposed some years ago. It is known [3]-[6] that the classical FFT is a special case of the FRFT. Fourier space and spatial frequency domains are special cases of fractional Fourier domains. They correspond to the αth fractional Fourier domains ($\alpha = 0$ and $\alpha = 1$, respectively).

In 1937, Condon wrote a paper called "Immersion of the Fourier transform in a continuous group of functional transformation" [1]. In 1961, Bargmann extended the FRFT in his paper [2], in which he gave definition of the FRFT, one based on Hermite polynomials as an integral transformation. If $H_n(\sqrt{2\pi}t)$ is a Hermite polynomial of order n then the functions

$$\Psi_n(t) = \frac{2^{1/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi}t)\exp(-\pi t^2)$$

for $n = 0, 1, 2, \ldots$ are eigenfunctions of the Fourier transform

$$\mathcal{F}[\Psi_n(t)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Psi_n(t)e^{2\pi t\tau}dt = \lambda_n\Psi_n(t),$$

with $\lambda_n = i^n$ being the eigenvalue corresponding to the nth eigenfunction and they form the orthonormal set of functions:

$$\int_{-\infty}^{+\infty} e^{-2\pi t^2}\frac{2^{1/4}}{2^n n!}H_n(\sqrt{2\pi}t)\frac{2^{1/4}}{2^m m!}H_m(\sqrt{2\pi}t)dt = \delta_{mn}. \tag{1}$$

According to Bargmann the FRFT $\mathcal{F}^\alpha := [\mathcal{F}^\alpha(\omega, t)]$ of order α may be defined through its eigenfunctions

$$\mathcal{F}^\alpha(\omega, t) := \sum_{n=0}^{\infty}\Psi_n(\omega)\Psi_n(t) =$$

$$= \sqrt{2}e^{[-i\pi(\omega^2+t^2)]}\sum_{n=0}^{\infty}\lambda_n^\alpha H_n(\sqrt{2\pi}\omega)H_n(\sqrt{2\pi}t),$$

where $\mathcal{F}^\alpha(\omega, t)$ is the kernel of the FRFT.

Obviously, the functions $\Psi_n(t)$ are the eigenfunctions of the FRFT

$$\mathcal{F}^\alpha[\Psi_n(t)] = \lambda_n^\alpha\Psi_n(t),$$

corresponding to the nth eigenvalues λ_n^α.

Of course for $\alpha = 1$ we have $\mathcal{F}^1(\omega, t) = e^{i\omega t}$. If $0 < |\alpha| < 2$ and $\alpha := 2\varphi/\pi$, then

$$\mathcal{F}^1(\omega, t) = \frac{\exp\left[-i\left(\frac{\pi\cdot\mathrm{sgn}(\sin\varphi)}{4} - \frac{\varphi}{2}\right)\right]}{\sqrt{\varphi}}\times$$

$$\times\exp\left[i\left(\pi\frac{\omega^2 - 2\omega t\cos\varphi + t^2}{\sin\varphi}\right)\right].$$

In 1980, Namias reinvented the FRFT again in his paper [3]. This approach was extended by McBride and

Kerr [4]. The FRFT was restricted to pure mathematical purposes. Very few publications appeared. Then Mendlovic and Ozaktas introduced the FRFT into the field of optics [5] in 1993. Afterwards, Lohmann [6] reinvented the FRFT based on the Wigner distribution function and opened the FRFT to optics applications. The Wigner distribution of a function $f(t)$ is defined as

$$W_f(t,\omega) := \int f(t+\frac{\tau}{2})f^*(t+\frac{\tau}{2})\exp(-2i\pi\tau\omega)d\tau.$$

There is the following relationship between the fractional FRFT and the Wigner distribution of a function $f(t)$:

$$W_{\mathcal{F}^\alpha[f]}(t,\omega) = W_f(t\ \cos\varphi - \omega\ \sin\varphi, t\ \sin\varphi + \omega\cos\varphi),$$

i.e. the FRFT is a rotation operation applied over the Wigner plane. Lohmann has proposed in [6] this relationship as the definition of FRFT.

In this paper we briefly introduce the multi-parametric FRFT and develop corresponding fast algorithm.

2. MULTI-PARAMETRIC FRACTIONAL FOURIER TRANSFORM

Discrete Fourier transform (DFT) $\mathcal{F}$ of length N is defined by

$$F(k) := \frac{1}{N}\sum_{n=0}^{N-1} f(n)e^{\frac{2\pi}{N}nk},$$

where $f(n)$ is the signal of the length N from the signal vector space $\mathbf{V}_N(e_0,e_1,\ldots,e_{N-1})$, spanned on the natural basis $e_0,e_1,\ldots,e_{N-1}$. In operator notation we write $\mathbf{F}=\mathcal{F}\mathbf{f}$. DFT has characteristic equation $\lambda^4=1$ since $\mathcal{F}^4=I$, where I is the identity operator. Consequently, the DFT $\mathcal{F}$ has only four eigenvalues in the form of solutions equations $\lambda^4=1:\lambda(k)=e^{j\frac{2\pi}{4}k}$, $k=0,1,2,3$. If $N=2^n$, then these eigenvalues have multiplicities $2^{n-2}+1$, $2^{n-2}-1$, 2^{n-2}, 2^n-2, respectively.

The Hermite polynomials $H_n(\sqrt{2\pi}t)$ (but not $\Psi_n(t)$) form a set that is orthonormal with respect to the weight function

$$w(t) = \exp(-2\pi t^2) = \exp\left(-\frac{t}{\sqrt{\frac{1}{2\pi}}}\right)^2.$$

It is well known that the discrete counterpart of a Gaussian window is a binomial window, i.e.

$$w(i) = \frac{1}{2^N}C_N^i$$

for $i=0,1,\ldots,N$. The (discrete) orthonormal polynomials that are associated with this window are known as Krawtchouk's polynomials

$$K_n(i) = \sum_{k=0}^{n}(-1)^{n-k}C_{N-i}^{n-k}C_i^k$$

for $i,n=0,1,\ldots,N$, i.e.

$$\sum_{i=0}^{n} C_N^i\left[\frac{1}{\sqrt{2^N C_N^n}}K_n(i)\right]\left[\frac{1}{\sqrt{2^N C_N^m}}K_m(i)\right] = \delta_{nm}.$$

The functions $\psi_n(i) := \sqrt{\frac{C_N^i}{2^N C_N^n}K_n(i)}$ form the set of the eigenvectors of the DFT:

$$\mathcal{F}[\psi_n(i)] = \lambda_n\psi_n(t).$$

For large values of N, the binomial window reduces to a Gaussian window. More specifically,

$$\lim_{N\to\infty}\frac{1}{2^N}C_N^{t+(N/2)} =$$

$$= \frac{1}{\sqrt{\pi N/2}}\exp\left[-\left(\frac{t}{\sqrt{N/2}}\right)^2\right]$$

for $t=-(N/2),\ldots,N/2$. It can be shown that the same limiting process turns a Krawtchouk polynomial into a Hermite polynomial, i.e.

$$\lim_{N\to\infty}K_n\left(t+\frac{N}{2}\right) = \frac{1}{\sqrt{2^n n!}}H_n\left(\frac{t}{\sqrt{N/2}}\right)$$

Hence, the discrete Hermite transform of length N approximates the analog Hermite transform of spread $\sigma=\sqrt{N/2}$.

Let $\mathbf{U}=[\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_{N-1}]$ be the matrix of eigenvectors of an discrete Fourier transform $\mathcal{F}$, then

$$\mathbf{U}\mathcal{F}\mathbf{U}^{-1} = \text{diag}\{\lambda(k)\}$$

and

$$\mathcal{F} = \mathbf{U}^{-1}\text{diag}\{\lambda(k)\}\mathbf{U}.$$

Definition 1 Let $\alpha_0,\alpha_1,\ldots,\alpha_{N-1}$ be arbitrary real numbers from $[0,1]$, then

$$\mathcal{F}^{\alpha_0,\alpha_1,\ldots,\alpha_{N-1}} :=$$

$$= \mathbf{U}^{-1}\left\{\text{diag}(\lambda_0^{\alpha_0}(k),\lambda_1^{\alpha_1}(k),\ldots,\lambda_{N-1}^{\alpha_{N-1}}(k))\right\}\mathbf{U} \quad (2)$$

is called *multi-parametric fractional Fourier transform* (MFRFT).

The set of all multi-parametric fractional $\mathcal{F}$-transforms form an abelian group $(\mathbf{R}/4) \times (\mathbf{R}/4) \times \ldots \times (\mathbf{R}/4)$, since

$$\mathcal{F}^{\alpha_0,\alpha_1,\ldots,\alpha_{N-1}}\mathcal{F}^{\beta_0,\beta_1,\ldots,\beta_{N-1}} =$$
$$= \mathcal{F}^{\alpha_0\oplus\beta_0,\alpha_1\oplus\beta_1,\ldots,\alpha_{N-1}\oplus\beta_{N-1}},$$

where $\oplus$ is the symbol of addition modulo 1. If $\alpha_i = \alpha$, $\forall i = 0, 1, \ldots, N-1$, then $\mathcal{F}^{\alpha_0,\alpha_1,\ldots,\alpha_{N-1}} = \mathcal{F}^{\alpha}$ is the classical fractional Fourier transform.

According to definition 1 efficient calculation of (1) require fast computational algorithms for transformation by $\mathbf{U}$ matrix ($\mathbf{U}$-transform).

3. FAST U-TRANSFORM FOR DFT

Let $\mathbf{Rot}\Big[\varphi_m \mid k_m, l_m\Big] := \left[\begin{array}{cc} \cos\varphi_m & \sin\varphi_m \\ -\sin\varphi_m & \cos\varphi_m \end{array}\right]_{k_m,l_m}$ be elementary Jacobi-Givens rotations in the 2D coordinate plane (e_{k_m}, e_{k_l}) of the signal space

$$\mathbf{V}(e_0, e_1, ..., e_{N-1}).$$

In this paper we use a sequential method for reduction of the classical Fourier transform using a finite sequence of elementary Jacobi-Givens rotations:

$$\mathcal{F}_{(m)} :=$$
$$= \mathbf{Rot}\Big[+\varphi_m \mid k_m, l_m\Big] \cdot \mathcal{F}_{(m-1)} \cdot \mathbf{Rot}\Big[-\varphi_m \mid k_m, l_m\Big],$$

where $m = 0, 1, ..., S$, $\mathcal{F}_{(0)} := \mathcal{F}$, $\mathcal{F}_{(S)} = \mathrm{diag}\{\lambda(k)\}$. The angles φ_m are determined so that $w^{(m)}_{k_m,l_m} = w^{(m)}_{l_m,k_m} = 0$, where $\mathcal{F}_{(m)} := \Big[w^{(m)}_{k_m,l_m}\Big]$.

The matrix $\mathcal{F}$ (without the first column and first row) is centro-symmetric. Therefore, it is block-diagonalized by $\frac{N}{2} - 1$ rotations of matrix $\mathbf{X}^{\oplus}_{0,N} :=$

$$\left(\sqrt{2}\mathbf{Rot}\left[0^o \mid 0, \frac{N}{2}\right] \prod_{i=1}^{\frac{N}{2}-1} \frac{2}{\sqrt{2}}\mathbf{Rot}\left[\frac{\pi}{4} \mid i, N-i\right]\right):$$

$$\mathcal{F}_{(\frac{N}{2}-1)} = \mathbf{X}^{\oplus}_{0,N}\mathcal{F}\mathbf{X}^{\oplus}_{0,N} = \Delta\mathbf{X}^{\oplus}_{0,N}\left[\mathbf{C}^{\oplus}_{0,N} \oplus \overline{I}\mathbf{S}\overline{I}\right]\mathbf{X}^{\oplus}_{0,N},$$

where Δ is a diagonal matrix, consisting of only $+1$ and -1, $\mathbf{C}_{\frac{N}{2}+1}$, $\mathbf{S}_{\frac{N}{2}-1}$ are discrete cosine and sine transforms, respectively, and $\overline{I}$ is the antidiagonal matrix.

The transforms $\mathbf{C}_{\frac{N}{2}+1}$ and $\mathbf{S}_{\frac{N}{2}-1}$ are centro-symmetric and they are block-diagonalized by $\frac{N}{2} - 1$ rotations of the matrix $\mathbf{X}^{\oplus}_{1,\frac{N}{2}-1} = \mathbf{X}^{\oplus}_{0,\frac{N}{4}+1} \oplus \mathbf{X}^{\oplus}_{1,\frac{N}{4}-1}$. After $N-2$ rotations we have the matrix

$$\mathcal{F}_{(N-2)} = \mathbf{X}^{\oplus}_{1,\frac{N}{2}-1}\mathcal{F}_{(\frac{N}{2}-1)}\mathbf{X}^{\oplus}_{1,\frac{N}{2}-1},$$

which is reducible to a block-diagonal form.

Example 1 Let $N = 8$ then

$$\mathcal{F}_{(3)} = \mathbf{X}^{\oplus}_{(0,8)}\mathcal{F}\mathbf{X}^{\oplus}_{(0,8)} =$$

$$= \frac{\Delta}{\sqrt{8}}\left(\left[\begin{array}{ccccc} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & 1 \\ \sqrt{2} & \sqrt{2} & \cdot & -\sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \cdot & -2 & \cdot & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & \cdot & \sqrt{2} & -\sqrt{2} \\ 1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & 1 \end{array}\right] \oplus\right.$$
$$\left.\oplus \left[\begin{array}{ccc} -\sqrt{2} & 2 & -\sqrt{2} \\ 2 & 2 & -2 \\ -\sqrt{2} & -2 & -\sqrt{2} \end{array}\right]\right)$$

and

$$\mathcal{F}_{(6)} = \mathbf{X}^{\oplus}_{(1,8)}\mathcal{F}\mathbf{X}^{\oplus}_{(1,8)} =$$

$$= \frac{\Delta}{\sqrt{8}}\left(\left[\begin{array}{ccccc} 2 & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2\sqrt{2} \\ 2 & \cdot & -2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \sqrt{2} & \cdot \\ \cdot & -2\sqrt{2} & \cdot & \cdot & \cdot \end{array}\right] \oplus\right.$$
$$\left.\left[\begin{array}{ccc} -2\sqrt{2} & \cdot & \cdot \\ \cdot & \cdot & -2\sqrt{2} \\ \cdot & -2\sqrt{2} & \cdot \end{array}\right]\right).$$

If

$$\mathbf{T}^3_8 := \left[I_1 \oplus \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \oplus \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right] \oplus I_3\right]$$

is the permutation matrix, then

$$\mathcal{F}'_{(6)} = \mathbf{T}^3_8\mathcal{F}_{(6)}\mathbf{T}^3_8 =$$

$$= \frac{\Delta}{\sqrt{8}}\left[\left[\begin{array}{cc} 2 & 2 \\ 2 & -2 \end{array}\right] \oplus \left[\begin{array}{cc} 0 & -2\sqrt{2} \\ -2\sqrt{2} & 0 \end{array}\right] \oplus\right.$$
$$\left.\oplus 2\sqrt{2} \oplus (-2\sqrt{2}) \oplus \left[\begin{array}{cc} 0 & -\sqrt{2} \\ -2\sqrt{8} & 0 \end{array}\right]\right].$$

Thus, we have to do three rotations into the planes (e_1, e_2), (e_3, e_4), (e_7, e_8), in order to obtain a scalar-diagonal matrix $\mathcal{F}_9 \equiv \mathrm{diag}(1, -1, 1, -1, -j, j, -j)$. These rotations are $\mathbf{T}^4_8 =$

$$= \frac{\sqrt{2}}{2}\left[\sqrt{2}\left[\begin{array}{cc} c^1_3 & -s^1_3 \\ s^1_3 & c^1_3 \end{array}\right] \oplus \left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right] \oplus\right.$$
$$\left.\oplus\sqrt{2} \oplus \sqrt{\oplus}\left[\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right]\right]$$

Therefore,

$$\mathbf{U} := \mathbf{T}^4_8\mathbf{T}^3_8\mathbf{X}^{\oplus}_{(1,8)}\mathbf{X}^{\oplus}_{(0,8)} =$$

$$= \left[\begin{array}{cccccccc} C^1_3 & -S^1_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ S^1_3 & C^1_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +d & +d & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +d & -d & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +d & +d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +d & -d \end{array}\right].$$

$$\cdot \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} +1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & 1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix},$$

where $C_m^k := \cos(\frac{k\pi}{2^m})$, $S_m^k := \sin(\frac{k\pi}{2^m})$, $d = \frac{\sqrt{2}}{2}$.

Finally, we give the matrix representation of the fast transform for $N = 16$ in the appendix.

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5. APPENDIX

Finally, as an example, we give the matrix representation of the fast transform for $N = 16$:

$$\mathbf{U} := \mathbf{T}_{16}^5 \mathbf{T}_{16}^4 \mathbf{T}_{16}^3 \mathbf{X}_{(1,16)}^{\oplus} \mathbf{X}_{(0,16)}^{\oplus},$$

where

$$\mathbf{T}_{16}^5 =$$

$$= \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +d & +d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -d & +d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & C_3^1 & \cdot & S_3^1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & C_3^1 & \cdot & S_3^1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -S_3^1 & \cdot & C_3^1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -S_3^1 & \cdot & C_3^1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix} \oplus$$

$$\oplus \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & C_3^1 & \cdot & S_3^1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & C_3^1 & \cdot & S_3^1 & \cdot \\ \cdot & \cdot & -S_3^1 & \cdot & C_3^1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -S_3^1 & \cdot & C_3^1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix},$$

$$\mathbf{T}_{16}^4 =$$

$$= \begin{bmatrix} +d & \cdot & -d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +d & \cdot & +d & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +d & \cdot & \cdot & +d & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +d & -d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +d & +d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -d & \cdot & \cdot & +d & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & C_4^1 & S_4^1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -S_4^1 & C_4^1 \end{bmatrix} \oplus$$

$$\oplus \begin{bmatrix} C_4^3 & -S_4^3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ S_4^3 & C_4^3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +d & \cdot & \cdot & +d & \cdot \\ \cdot & \cdot & \cdot & +d & -d & \cdot & \cdot \\ \cdot & \cdot & \cdot & +d & +d & \cdot & \cdot \\ \cdot & \cdot & -d & \cdot & \cdot & +d & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{bmatrix},$$

$$\mathbf{T}_{16}^3 =$$

$$= \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \end{bmatrix} \oplus$$

$$\oplus \begin{bmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \end{bmatrix},$$

$$\mathbf{X}_{(1,16)}^{\oplus} =$$

$$
= \begin{bmatrix}
+1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & +1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & +1 & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot\cdot & \\
\cdot & \cdot & \cdot & +1 & \cdot & +1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & +1 & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\
\cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\
+1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1
\end{bmatrix} \oplus
$$

$$
\oplus \begin{bmatrix}
+1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & +1 & \cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & +1 & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot \\
\cdot & \cdot & +1\,t & \cdot & +1 & \cdot & \cdot \\
\cdot & +1 & \cdot & \cdot & \cdot & +1 & \cdot \\
+1 & \cdot & \cdot & \cdot & \cdot & \cdot & +1
\end{bmatrix},
$$

$$
\mathbf{X}^{\oplus}_{(0,16)} =
$$

$$
= \begin{bmatrix}
\sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\
\cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1
\end{bmatrix}
$$

Similar expressions were found for **U**-transform lengths up to 256.