TIME-FREQUENCY LOCALIZATION OF COMPACTLY SUPPORTED WAVELETS

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ABSTRACT

Two items will be presented in the talk. The first one is modification of classic Daubechies compactly supported wavelets preserving localization in time with the growth of smoothness. The second one is nonstationary wavelets with the use of scaling filters that vary from one scale to the next finer one.

Wavelets are a tool for decomposing functions in various applications. It is closely connected with transforms in signal processing (subband coding algorithms, multiresolution transform in computer vision). Localization in time and frequency of wavelets are of main significance for applications.

A function $\psi \in L^2(\mathbf{R})$ is called a stationary orthonormal wavelet if its normalized, translated dilates $\psi_{jk}(t) := 2^{j/2}\psi(2^j t - k), \quad j,k \in \mathbf{Z} \quad t \in \mathbf{R}$, form an orthonormal basis for $L^2(\mathbf{R})$. In the wavelet theory the localization of φ is characterized by the radius of autocorrelation function $\Phi(t) := \int_{\mathbf{R}} \varphi(s)\varphi(s-t)\,ds$:

$$\Delta(\Phi) := \left\{ \int_{\mathbf{R}} t^2 \, |\Phi(t)|^2 dt \Big/ \int_{\mathbf{R}} |\Phi(t)|^2 dt \right\}^{1/2}.$$

Similar constant for Fourier transform

$$\widehat{\Phi}(\omega) := \int_{\mathbf{R}} \Phi(t) e^{-i\omega t}\, dt$$

is defined as

$$\Delta(\widehat{\Phi}) := \left\{ \int_{\mathbf{R}} \omega^2 \, |\widehat{\Phi}(\omega)|^2 d\omega \Big/ \int_{\mathbf{R}} |\widehat{\Phi}(\omega)|^2 \, d\omega \right\}^{1/2}.$$

The product of radii $\Delta(\Phi)\Delta(\widehat{\Phi})$ is called an uncertainty constant of Φ.

The uncertainty constants of classic Daubechies wavelets [1] run to infinity with the growth of smoothness, because they approximate in $L^2(\mathbf{R})$ the Shannon wavelets [2]. The Daubechies construction can be modified to obtain compactly supported wavelets approximating in $L^2(\mathbf{R})$ the Meyer wavelets, and as a consequence the uncertainty constants of that wavelets are uniformly bounded with respect to smoothness.

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1. CONSTRUCTION OF MODIFIED DAUBECHIES WAVELETS

Let $a \in (0,1)$ and $f_a(t)$ be infinitely differentiable nonnegative function on $[-1,1]$ equal to 0 if $t \in [-1,-a]$ and satisfying the identity $f_a(t) + f_a(-t) = 1$, $t \in [-1,1]$. Denote by $b_l^N(t) := \binom{N}{l}(\frac{1+t}{2})^l(\frac{1-t}{2})^{N-l}$, $l = 0,1,...,N$, the Bernstein polynomials for the interval $[-1,1]$, $\quad t_{N,l} := \frac{2l-N}{N},\ l = 0,1,...,N$.

For every $N \in \mathbf{N}$ define trigonometric polynomial $m_N^a(\omega)$ with real coefficients by the equation

$$|m_N^a(\xi)|^2 = B_N^a(\cos\xi), \quad m_N^a(0) = 1, \qquad (1)$$

where $B_N^a(t) = \sum_{l=0}^{N} f_a(t_{N,l}) b_l^N(t)$ are Bernstein polynomial approximating f_a. Such polynomial m_N^a exists by the F.Riece lemma.

Define the scaling function $\varphi^{a,N}$ and the corresponding wavelet $\psi^{a,N}$ by Fourier transform:

$$\widehat{\varphi}^{a,N}(\omega) = \prod_{l=1}^{\infty} m_N^a(\tfrac{\omega}{2^l}),$$
$$\widehat{\psi}^{a,N}(\omega) = e^{-i\omega/2}\overline{m_N^a(\tfrac{\omega}{2}+\pi)} \prod_{l=2}^{\infty} m_N^a(\tfrac{\omega}{2^l}).$$

Theorem 1.1 *[3] For every $N \in \mathbf{N}$ $\psi^{a,N}$ is a stationary orthonormal wavelet and there exists a constant $\mu > 0$ such that functions $\varphi^{a,N}$ and $\psi^{a,N}$ belong to $C^{\mu N}$, where*

$$C^{\alpha} := \{f : \int_{\mathbf{R}} \hat{f}(\omega)(1+|\omega|)^{\alpha} d\omega < \infty\},\ \alpha > 0.$$

Remark 1.1 *If in the above construction we take the characteristic function of $[0,1]$ instead of the function f_a we will obtain the classical Daubechies wavelets.*

2. UNCERTAINTY CONSTANTS OF MODIFIED DAUBECHIES WAVELETS

In order to estimate the uncertainty constants of the modified Daubechies wavelets we need the definition of Meyer scaling function [4] φ^M :

$$\widehat{\varphi^M}(\omega) := \sqrt{f_a(\cos(\omega/2))}\chi_{[-2\pi,2\pi]}(\omega),$$

which generates the wavelets with compactly supported Fourier transform. Here χ_e is a characteristic function of set e.

Denote by

$$\Phi^M(t) := \int_{\mathbf{R}} \varphi^M(s)\varphi^M(s-t)\,ds,$$
$$\Phi^{a,N}(t) := \int_{\mathbf{R}} \varphi^{a,N}(s)\varphi^{a,N}(s-t)\,ds.$$

the corresponding autocorrelation functions of Meyer scaling function and modified Daubechies scaling function.

Theorem 2.1 *If the function f_a satisfies conditions*

$$(f_a)''(t) \geq 0 \quad \textit{if} \quad t \leq 0, \quad (f_a)''(t) \leq 0 \quad \textit{if} \quad t > 0,$$
$$(f_a)'(t) \leq C\sqrt{f_a(t)} \quad \textit{for some constant} \quad C,$$

then

$$\lim_{N\to\infty} \|\widehat{\Phi}^{a,N} - \widehat{\Phi}^M\|_{L_p} = 0, \quad 1 \leq p < \infty;$$
$$\lim_{N\to\infty} \Delta_{\widehat{\Phi}^{a,N}} = \Delta_{\widehat{\Phi}^M};$$
$$\lim_{N\to\infty} \Delta_{\Phi^{a,N}} \leq C\frac{4\sqrt{2}}{(\sqrt{2}-1)\|\Phi^M\|_2};$$

and consequently

$$\lim_{N\to\infty} \Delta_{\widehat{\Phi}^{a,N}} \Delta_{\Phi^{a,N}} \leq \Delta_{\widehat{\Phi}^M} \frac{4\sqrt{2}}{(\sqrt{2}-1)\|\Phi^M\|_2} < \infty.$$

3. NONSTATIONARY WAVELETS

The concept of nonstationary orthogonal wavelets means that functions ψ_{jk}, while still being assumed to be the 2^{-j}-shifts of functions ψ_{j0}, are not be assumed to be the dilates of ψ_{0k}. This concept was introduced in [5],[6]. It turns out that this generalization leads to interesting bases for $L^2(\mathbf{R})$. For example, it is proved in [4] and [8] that there does not exist stationary orthogonal infinitely differentiable compactly supported wavelet. However, it is shown in [6] and [7] how to construct nonstationary orthonormal infinitely differentiable compactly supported wavelets. Nonstationary scaling functions are defined in Fourier domain by equations

$$\widehat{\varphi}_j^T(\omega) := 2^{-j/2}\prod_{l=j+1}^{\infty} m_{T(l)}(\omega 2^{-l}),$$
$$\widehat{\varphi}_j^{a,T}(\omega) := 2^{-j/2}\prod_{l=j+1}^{\infty} m^a_{T(l)}(\omega 2^{-l}),$$

where $T := \{T(N)\}_{N\in\mathbf{N}}$ is a sequence of positive integers satisfying the conditions

$$\lim_{N\to\infty} T(N) = \infty, \quad T(N) \leq T(N+1) \leq T(N)+1,$$

m_N is classical Daubechies filters (Remark 1.1), m_N^a is modified Daubechies filters defined by (1). As usual

$$\varphi^T_{j,k}(t) := \varphi^T_j(t-k2^{-j}),$$
$$\varphi^{a,T}_{j,k}(t) := \varphi^{a,T}_j(t-k2^{-j}).$$

The corresponding wavelets are defined by

$$\widehat{\psi}^T_j(\omega) := e^{-i\omega 2^{-j-1}} m_{T(j+1)}(-\omega 2^{-j-1}-\pi)\widehat{\varphi}^T_{j+1},$$
$$\widehat{\psi}^{a,T}_j(\omega) := e^{-i\omega 2^{-j-1}} m^a_{T(j+1)}(-\omega 2^{-j-1}-\pi)\widehat{\varphi}^{a,T}_{j+1},$$
$$\psi^T_{j,k}(t) = \psi^T_j(t-k2^{-j}), \quad \psi^{a,T}_{j,k}(t) = \psi^{a,T}_j(t-k2^{-j}).$$

Theorem 3.1 *Systems*

$$\Psi^T := \{\varphi^T_{0k}, \psi^T_{jk} \quad j,k \in Z, \quad j \geq 0\},$$

$$\Psi^{a,T} := \{\varphi^{a,T}_{0k}, \psi^{a,T}_{jk} \quad j,k \in Z, \quad j \geq 0\}$$

have the following properties:

Ψ^T, $\Psi^{a,T}$ *are orthonormal bases for* $L^2(\mathbf{R})$;
$\Psi^T \subset C^\infty(R)$; $\Psi^{a,T} \subset C^\infty(R)$;
$\operatorname{supp} \varphi^T_{00} \subset [0, 2T(1)-1]$;
$\operatorname{supp} \varphi^{a,T}_{00} \subset [0, 2T(1)-1]$;
$\operatorname{supp} \psi^T_{j0} \subset [-(T(j+2)+1)2^{-j}, (T(j+1)-1)2^{-j}]$,
$\operatorname{supp} \psi^{a,T}_{j0} \subset [-(T(j+2)+1)2^{-j}, (T(j+1)-1)2^{-j}]$
$j \in \mathbf{Z}$, $j \geq 0$.

For Ψ^T Theorem 3.1 is proved in [7], for $\Psi^{a,T}$ the reasoning is completely similar. A general theory of nonstationary multiresolution analysis with trigonometric filters is presented in [9].

4. UNCERTAINTY CONSTANTS OF NONSTATIONARY WAVELETS

The drawback of nonstationary wavelets Ψ^T constructed on the basis of classic Daubechies filters is the fact that their uncertainty constants are not uniformly bounded.

Since sequence T is fixed through this paper we omit this index to simplify the notation. Let Φ_j be autocorrelation function of φ^T_j. It is more convenient to estimate uncertainty constants of autocorrelation functions being dilated to the zero scale: $\Phi^0_j(t) := \Phi_j(2^{-j}t)$.

Theorem 4.1

$$\lim_{j\to\infty} \|\widehat{\Phi^0_j} - \chi_{[-\pi,\pi]}\|_{L_p} = 0, \quad 1 \leq p < \infty,$$
$$\liminf_{j\to\infty} \Delta(\widehat{\Phi^0_j}) \geq \frac{\pi}{\sqrt{3}};$$
$$\lim_{j\to\infty} \Delta(\Phi^0_j) = \infty;$$

and consequently

$$\lim_{j\to\infty} \Delta(\widehat{\Phi_j^0})\Delta(\Phi_j^0) = \infty.$$

However, the application of modified Daubechies filters instead of classical ones in the construction of nonstationary wavelets leads to nonstationary orthonormal infinitely differentiable compactly supported wavelets with uniformly bounded uncertainty constants.

Let Φ_j^a be autocorrelation function of $\varphi_j^{a,T}$. We again shall estimate uncertainty constants of autocorrelation functions dilated to the zero scale: $\Phi_j^{a,0}(t) := \Phi_j^a(2^{-j}t)$.

Theorem 4.2 *If function f_a satisfies conditions*

$$(f_a)''(t) \geq 0 \quad \text{if} \quad t \leq 0, \quad (f_a)''(t) \leq 0 \quad \text{if} \quad t > 0,$$
$$(f_a)'(t) \leq C\sqrt{f_a(t)} \quad \text{for some constant} \quad C,$$

then

$$\begin{aligned} &\lim_{j\to\infty} \|\widehat{\Phi_j^{a,0}} - \widehat{\Phi}^M\|_{L_p} = 0, \quad 1 \leq p < \infty; \\ &\lim_{j\to\infty} \Delta(\widehat{\Phi_j^{a,0}}) = \Delta(\widehat{\Phi}^M); \\ &\lim_{j\to\infty} \Delta(\Phi_j^{a,0}) \leq C\frac{4\sqrt{2}}{(\sqrt{2}-1)\|\Phi^M\|_2}; \end{aligned}$$

and consequently

$$\lim_{j\to\infty} \Delta(\widehat{\Phi_j^{a,0}})\Delta(\Phi_j^{a,0}) \leq \frac{4\sqrt{2}}{(\sqrt{2}-1)\|\Phi^M\|_2}\Delta(\widehat{\Phi}^M) < \infty.$$

The numerical implementation of nonstationary wavelet analysis has the same pyramidal structure as in the standard fast wavelet transform algorithm, using longer filters at the high scales and smaller filters at the coarse scale. The idea is very natural in terms of signal processing since it does not make sense to use a filter with a comparable size to that of the whole signal at the coarse scales. The regularity of the construction will not be affected by the use of small filters (corresponding to nonregular scaling functions) at the coarsest scales.

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