FAST BILINEAR ALGORITHMS FOR IMAGE COMPRESSION AND RECONSTRUCTION WITH SUPER-RESOLUTION

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ABSTRACT

A wide class of fast image compression algorithms are elaborated in this paper. Proposed technique differ from the known spectral algorithms so that it truncates generalized cepstrum rather than spectrum together with additional information on spectrum phase for increasing of compression rate.

1. INTRODUCTION

Modern video information registration and transmission systems are characterized by high rate of input information flow and large volumes of accumulated information. Such situation generates serious problems especially when data are transmitted in real time and in a limited frequency band. Hence a problem of reduction of redundancy of the processed space-time signals is very practically actual.

The source of statistical redundancy of an image is a high correlation of its samples. For its reduction $(N \times N)$ -sample image \mathbf{s}_N is transformed using 2-D unitary transform (UT)

$$\mathbf{S}_N = \mathcal{F}\mathbf{s}_N$$

which partially decorrelates the samples of the image. The essence of *linear compression* is the following: the most "powerful" $L^2 < N^2$ spectral samples

$$\mathbf{S}_L = \chi(\Omega) \mathbf{S}_N$$

are transmitted through the telemetric channel (instead of N^2 initial samples), where $\chi(\Omega)$ is characteristic function of transparency $(L \times L)$ -window.

Image estimation is made in reverse order:

$$\widehat{\mathbf{s}}_N = \mathcal{F}^{-1} \mathbf{S}_L.$$

This algorithm of compression and reconstruction is sufficiently well studied and investigated now for a wide class of unitary transforms: Fourier, Walsh transforms, cosine, sine transforms etc. They are called *linear spectral algorithms* of compression.

The total decorrelation (and hence maximal compression) is achieved by Karhunen–Loeve transform. But KLT has no fast algorithm. Hence in practice it is changed by a unitary transforms having fast algorithms. The gain of speed is accompanied with the loss of quality of the reconstructed image.

To overcome this shortcoming (especially for wideband images) the present paper suggest

- fast bilinear generalized auto-correlation and cepstrum compression algorithms,
- methods of extrapolation of the received truncated spectrum \mathbf{S}_L , that would allow to realize fast linear and bilinear algorithms of *reconstruction with super-resolution* of compression image.

2. FAST BILINEAR GENERALIZED AUTO-CORRELATION AND CEPSTRUM COMPRESSION ALGORITHMS

Let $\mathbf{K} := [0, N-1] \times [0, N-1]$ and $\mathbf{K}^* := [0, N-1] \times [0, N-1]$ are two examples of rectangular $(N \times N)$ -point subset of the set $\mathbf{Z}^2 := \mathbf{Z} \times \mathbf{Z}$. The first of them will be called a space (spatial) domain and the second one - - a \mathcal{F} -spectrum domain. Let us consider images as functions of brightness of the form

$$\mathbf{s}_N(n): \mathbf{K} \longrightarrow \mathbf{R}^+,$$

where $n := (n_1, n_2) \in \mathbf{K} \subset \mathbf{Z}^2$, \mathbf{R}^+ is the set of positive numbers.

The space of images will be denoted as

$$\mathbf{L}_{2}(\mathbf{K},\mathbf{R}^{+}):=\left\{\mathbf{s}_{N}\mid\mathbf{s}_{N}:\mathbf{K}\longrightarrow\mathbf{R}^{+}\right\}$$

We introduce an arbitrary orthogonal basis

$$\phi_{\alpha}(n) \in \mathbf{L}_2(\mathbf{K}, \mathbf{R}^+),$$

where $n := (n_1, n_2) \in \mathbf{K}$, $\alpha := (\alpha_1, \alpha_2) \in \mathbf{K}^*$. Then any image can be decomposed in generalized *Fourier* series in this basis

$$\mathbf{s}_N(n) = \sum_{\alpha \in \mathbf{K}^*} \mathbf{S}_{\mathbf{N}}(\alpha) \phi_\alpha(n)$$

where

$$\mathbf{S}_N(\alpha) = \sum_{n \in \mathbf{K}} \mathbf{s}_N(\alpha) \overline{\phi}_{\alpha}(n),$$

or in matrix notation

$$\mathbf{s}_N = \mathcal{F}^{-1} \mathbf{S}, \quad \mathbf{S}_N = \mathcal{F} \mathbf{s}.$$

This expressions are called *generalized inverse and di*rect Fourier transforms.

Definition 1 Image \mathbf{s}_{L}^{\bullet} that takes nonzero values in some $(L \times L)$ -point subset $\Omega \in \mathbf{K}$ of the space domain \mathbf{K} will be called *image with finite support* (= *finite image*).

Definition 2 Image \mathbf{s}_N° the spectrum **S** of which takes nonzero values in some $(L^* \times L^*)$ -point subset $\Omega^* \in \mathbf{K}^*$ of the spectrum domain \mathbf{K}^* will be called *image with finite spectrum*.

Definition 3 Generalized \mathcal{F} -auto-correlation function $(\mathcal{F}$ -ACF) of the image \mathbf{s}_N is called an expression

$$\operatorname{COR}_{N}\{\mathbf{s}_{N}\} := \mathcal{F}^{-1}\left\{ |\mathcal{F}\{\mathbf{s}_{N}\}|^{2} \right\} = \mathcal{F}^{-1}\left\{ |\mathbf{S}_{N}|^{2} \right\}$$

If the \mathcal{F} is Fourier transform then COR is classical (arithmetic) auto-correlation function, if the \mathcal{F} is Walsh transform then COR is dyadic (logical) auto-correlation function etc.

Definition 4 Generalized (\mathcal{F}, f) -cepstrum of image \mathbf{s}_N is called an expression

$$\operatorname{CEP}_{N}\{\mathbf{s}_{N}\} := \mathcal{F}^{-1}\{f\left(\mathcal{F}\{\mathbf{s}_{N}\}\right)\}$$

Surely, in the case when pair (\mathcal{F}, f) is the Fourier transform and a square, then CEP = COR, if it is Fourier transform and $f = \log$, then CEP is classical cepstrum. In all another cases we obtained a generalized auto-correlation functions or a generalized cepstrum.

Definition 5 Image \mathbf{s}_N the cepstrum $\operatorname{CEP}_L\{\mathbf{s}_N\}$ of which takes nonzero values in some $(L \times L)$ -point subset $\Omega \in \mathbf{K}$ of the space domain \mathbf{K} will be called *image with finite cepstrum*.

Let us suppose that f is whitening function (type log, $\sqrt{}$, etc), that is, a function equalizing a spectrum in the spectral domain of generalized frequencies Ω^* .

In this case $\operatorname{CEP}_N\{\mathbf{s}\}$ (or $\operatorname{COR}_N\{\mathbf{s}\}$ as special case of CEP) will be of a *small localization*, despite of the initial signal spectrum \mathbf{S}_N width. This fact give a possible to truncate CEP_L to $L^2 << N^2$ samples (where $L^2 := \operatorname{Card}(\Omega^*)$ is cardinality of the subset Ω^*) and their transition along the communication channels. This allow obtain the compression coefficient $K_{comp} = N/L$. At the receiving side reconstruction is realized in the inverse order:

$$\widehat{\mathbf{s}}_N = \mathcal{F}^{-1} \left\{ f^{-1} \left\{ \mathcal{F} \{ \mathbf{CEP}_L(\mathbf{s}) \} \right\} \right\}$$

if the function f doesn't destruct information about the spectrum phase. In opposite case it must be compensated by the transmission of the phase $\Phi_N := \arg(\mathbf{S}_N)$ along the communication channel together with $\operatorname{CEP}_L(\mathbf{S})$. In this case the reconstruction algorithm becomes of the form:

$$\widehat{\mathbf{s}}_N = \mathcal{F}^{-1} \left\{ \Phi_N \cdot f^{-1} \left\{ \mathcal{F} \{ CEP_L(\mathbf{s}) \} \right\} \right\}$$

In particular, when $f = f^{-1} = 1$, and \mathcal{F} is orthogonal transform, we obtain the image compression and reconstruction algorithm basing on image \mathcal{F} -spectrum phase:

$$\widehat{\mathbf{s}}_N = \mathcal{F}^{-1} \{ \Phi_N \} = \mathcal{F}^{-1} \{ \operatorname{sign}(\mathbf{S}_N) \}$$

that asks for compression one bit per one pixel of image.

3. GENERALIZED FUNCTIONS WITH DOUBLE ORTHOGONALITY

In [2], Slepian addressed the quation of the extent to which a signal can be simultaneously concentrated in both time and frequency. The answer is related to the linear prolate spheroidal functions (LPSf's). They maximize the proportion

$$\lambda = \frac{\int\limits_{-\Omega}^{+\Omega} |X(f)|^2 df}{\int\limits_{-\infty}^{+\infty} |X(f)|^2 df}$$

where

$$X(f) := \int_{-T}^{+T} x(t) e^{-j\omega t} dt.$$

LPSF's $\Phi_n(c, t)$ are a set of bandlimited functions constructed to be invariant to the Fourier transform and orthogonal on the real line for the given bandwidth Ω :

$$\int_{-\infty}^{+\infty} \Phi_n(c,t) e^{-j\omega t} dt =$$

$$= \begin{cases} i^{-n} \sqrt{\frac{2\pi T}{\Omega \lambda_n}} \Phi_n(c, \omega T/\Omega), & |\omega| \le \Omega, \\ 0, & |\omega| \ge \Omega, \end{cases}$$
$$\int_{-\infty}^{+\infty} \Phi_n(c, t) \Phi_k(c, t) dt = \begin{cases} 1, & n = k, \\ 0, & n \ne k \end{cases}$$

and simultaneously to be invariant to the Fourier transform and orthogonal over a finite interval [+T, -T]:

$$\int_{-T}^{T} \frac{\sin \Omega(t-\tau)}{\pi(t-\tau)} \Phi_n(c,\tau) d\tau = \lambda_n \Phi_n(c,t),$$
$$\int_{-T}^{+T} Phi_n(c,t) \Phi_k(c,t) dt = \begin{cases} \lambda_n, & n=k, \\ 0, & n\neq k, \end{cases}$$

where $\Phi_n(c, t)$ denotes the LPSF of order *n* and the number of degrees of freedom $c = T\Omega$, $\lambda_n(c)$ are the (integral) linear prolate eigenvalues $1 > \lambda_o(c) > \lambda_1(c) > \cdots$.

These unique properties make LPSF's useful insignal processing. In particular, any square–integrable function f(t) of bandwidth Ω known in the basic interval [-T,T] can be extrapolated to infinity by the series

$$f(t) = \sum_{n=0}^{\infty} a_n(c)\Phi_n(c,t), \quad t \in (-\infty, +\infty)$$

where $a_n(c)$ are calculated from f(t) in the basic interval

$$a_n(c) := \frac{1}{\lambda_n(c)} \int_{-T}^{+T} f(t) \Phi_n(c,t) dt.$$

This approach is widely used. e.g., to achieve superresolution in optical image reconstruction [5],[6].

Let $\chi(\Omega)$ and $\chi(\Omega^*)$ are characteristic functions of the subsets Ω and Ω^* , respectively and let

$$\mathcal{X} := \operatorname{diag}(\chi(\Omega)), \quad \mathcal{X}^* := \operatorname{diag}(\chi(\Omega^*))$$

are $(N \times N)$ -diagonal matrices.

Definition 6 Operators

$$\mathcal{F}_o := \mathcal{XFX}^*, \quad \mathcal{F}_o^+ := \mathcal{X}^* \mathcal{F}^{-1} \mathcal{X}$$

will be called *truncated direct and inverse generalized* Fourier transforms.

They belong to the class Hilbert–Shmidt operators. Let

$$\mu_l^{\bullet}(n), \ n \in \Omega, \quad \eta_k^{\circ}(\alpha), \ \alpha \in \Omega^*,$$

(where $l = 0, 1, 2, ..., L - 1, k = 0, 1, ..., L^* - 1$) are eigen-functions of the operators $\mathcal{F}_o^+ \mathcal{F}_o, \mathcal{F}_o \mathcal{F}_o^+$:

$$\left(\mathcal{F}_{o}^{+}\mathcal{F}_{o}\right)\mu_{l}^{\bullet}(n) = \rho_{l}\mu_{l}^{\bullet}(n), \quad \left(\mathcal{F}_{o}\mathcal{F}_{o}^{+}\right)\eta_{l}^{\circ}(\alpha) = \rho_{l}\eta_{l}^{\circ}(\alpha)$$

that correspond to the eigenvalues ρ_l . From this expressions we have

$$\mathcal{F}_{o}^{+}\mathcal{F}_{o} = \sum_{l=0} \rho_{l} \mu_{l}^{\bullet}(n) \mu_{l}^{\bullet}(m),$$
$$\mathcal{F}_{o}\mathcal{F}_{o}^{+} = \sum_{l=0} \rho_{l} \eta_{l}^{\circ}(\alpha) \eta_{l}^{\circ}(\beta),$$

or in matrix form

$$\mathcal{F}_o^+ \mathcal{F}_o = \mathcal{M}^{\bullet} \Lambda \mathcal{M}^{\bullet t}, \quad \mathcal{F}_o \mathcal{F}_o^+ = \mathcal{N}^{\circ} \Lambda \mathcal{N}^{\circ t},$$

where $\mathcal{M}^{\bullet} := [\mu_l^{\bullet}(n)], \mathcal{N}^{\circ} := [\nu_k^{\circ}(\alpha)].$ Obviously,

$$\mathcal{F}_0 = \sum_{l=0} \rho^{1/2} \mu_l^{\bullet}(n) \eta_l^{\circ}(\alpha),$$
$$\mathcal{F}_0^+ = \sum_{l=0} \rho^{1/2} \eta_k^{\circ}(\alpha) \mu_k^{\bullet}(n),$$

or in matrix form

$$\mathcal{F}_o = \mathcal{N}^{\circ} \Lambda^{1/2} \mathcal{M}^{\bullet t}, \quad \mathcal{F}_o^+ = \mathcal{M}^{\bullet} \Lambda^{1/2} \mathcal{N}^{\circ t}$$

are singular decompositions of transforms \mathcal{F}_o and \mathcal{F}_0^+ , respectively

Note that the supports of $\mu_l(n)$, $\eta_l(\alpha)$ are the transparency $(l \times L)$ - and $(L^* \times L^*)$ -windows Ω, Ω^* .

Consider new function systems:

$$\mu_l(n) := \mathcal{F}^{-1} \eta_l^{\circ}(\alpha), \quad \eta_l(\alpha) := \mathcal{F} \mu_l^{\bullet}(n).$$

Domains \mathbf{K} and \mathbf{K}^* are their supports. Transforms operators in the basis

$$\mu_l(n), \ n \in \Omega, \quad \eta_l(\alpha), \ \alpha \in \Omega^*,$$

will be denoted as \mathcal{M} and \mathcal{N} . Surely,

$$\mathcal{N} = \mathcal{F}\mathcal{M}^{\bullet}, \quad \mathcal{M} = \mathcal{F}^{-1}\mathcal{N}^{\circ}.$$

Theorem 1 Functions $\mu_l(n), \eta_l(\alpha)$ are orthonormalized on **K** and **K**^{*}:

$$\sum_{n \in \mathbf{K}} \mu_{l_1}(n) \mu_{l_2}(n) = \delta_{l_1 l_2}, \text{ or } \mathcal{M}\mathcal{M}^t = I,$$
$$\sum_{\alpha \in \mathbf{K}^*} \eta_{l_1}(\alpha) \eta_{l_2}(\alpha) = \delta_{l_1 l_2}, \text{ or } \mathcal{N}\mathcal{N}^t = I$$

and orthogonal on the transparency windows Ω, Ω^* :

$$\sum_{n\in\Omega} \mu_{l_1}(n)\mu_{l_2}(n) = \rho_{l_1}\delta_{l_1l_2}, \text{ or } \mathcal{MXM}^t = \Lambda,$$
$$\sum_{\alpha\in\Omega^*} \eta_{l_1}(\alpha)\eta_{l_2}(\alpha) = \rho_{l_1}\delta_{l_1l_2} \text{ or } \mathcal{NX^*N}^t = \Lambda,$$

where $\Lambda = \text{diag}(\rho_l)$, "t" is symbol of transpose.

Definition 7 Functions $\mu_l(n), \eta_l(\alpha)$ will be called generalized \mathcal{F} -functions with double orthogonality or Generalized Prolate Spheroidal Functions [2],[4]

It must be mentioned that

$$\chi(\Omega)\mu_l(n) = \rho^{1/2}\mu_l^{\bullet}(n), \chi(\Omega^*)\eta_l(\alpha) = \rho^{1/2}\eta_l^{\circ}(\alpha),$$

or in operator form

$$\mathcal{M}\mathcal{X} = \Lambda \mathcal{M}^{\bullet}, \quad \mathcal{N}\mathcal{X}^* = \Lambda \mathcal{N}^{\circ}.$$

4. RECONSTRUCTION WITH SUPER-RESOLUTION

Let \mathbf{s} be a signal with finite spectrum or finite cepstrum. Then estimations

$$\widehat{\mathbf{s}}_N = \mathcal{F}^{-1} \mathbf{S}_L, \quad \widehat{\mathbf{s}}_N = \mathcal{F}^{-1} \left\{ f^{-1} \left\{ \mathcal{F} \{ \mathbf{CEP}_L(\mathbf{s}) \} \right\} \right\}$$

contains information about the spectrum of initial image \mathbf{s}_N only in the transparency spectrum domain Ω^* and space domain Ω and has not information outside. For its reconstruction we extrapolate the truncated spectrum and cepstrum on the whole spectral \mathbf{K}^* and space \mathbf{K} domains, respectively, using the functions with double orthogonality

 and

$$\mathbf{S}_N^{ext} = \left(\mathcal{N}\Lambda^{-1}\mathcal{N}^t\right)\mathbf{S}_L,$$

$$\operatorname{CEP}_{N}^{ext} = \left(\mathcal{M}\Lambda^{-1}\mathcal{M}^{t}\right)\operatorname{CEP}_{L}.$$

Then we reconstruct the initial image from \mathbf{S}_N^{ext} or CEP_N^{ext}

$$\begin{aligned} \widehat{\mathbf{s}}_{N}^{sup} &= \mathcal{F}^{-1} \mathbf{S}_{N}^{ext} = \mathcal{F}^{-1} \left\{ \left(\mathcal{N} \Lambda^{-1} \mathcal{N}^{t} \right) \mathbf{S}_{L} \right\}, \\ \widehat{\mathbf{s}}_{N}^{sup} &= \mathcal{F}^{-1} \left\{ f^{-1} \left\{ \mathcal{F} \{ \mathbf{CEP}_{N}^{exp}(\mathbf{s}) \} \right\} \right\} = \\ &= \mathcal{F}^{-1} \left\{ f^{-1} \left\{ \mathcal{F} \{ \left(\mathcal{M} \Lambda^{-1} \mathcal{M}^{+} \right) \mathbf{CEP}_{L}(\mathbf{s}) \} \right\} \right\}. \end{aligned}$$

As an extrapolation reconstruction missing spectral and cepstral samples then the image $\hat{\mathbf{s}}_N^{sup}$ unlike $\hat{\mathbf{s}}_N$ is reconstructed including all its "fine details" that can be treated as *super-resolution effect*.

If noise there is in communication channel then the truncated and distorted spectrum $\mathbf{S}_l + \mathbf{N}_l$ appears at the receiving side. In this case reconstruction with supper-resolution must be realized in the basis $\eta_l(n)$ for spectral reconstruction and in the basis $\mu_l(\alpha)$ for cepstral reconstruction with using optimal Wiener filtration in these bases:

$$\widehat{\mathbf{s}}_{N}^{Win,sup} = \mathcal{F}^{-1} \left\{ \left(\mathcal{N} \Lambda_{spec}^{Wien} \mathcal{N}^{+} \right) \mathbf{S}_{L} \right\},$$
$$\widehat{\mathbf{s}}_{N}^{Win,sup} =$$
$$= \mathcal{F}^{-1} \left\{ f^{-1} \left\{ \mathcal{F} \left\{ \left(\mathcal{M} \Lambda_{cep}^{Wien} \mathcal{M}^{+} \right) \operatorname{CEP}_{L}(\mathbf{s}) \right\} \right\} \right\}$$

where Λ_{spec}^{Wien} , Λ_{cep}^{Wien} are transfer (frequency system) function of optimal Wiener filter in the bases $\eta_l(n)$ $\mu_l(\alpha)$, respectively.

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