A SUPERFAST CONVOLUTION TECHNIQUE FOR VOLTERRA FILTERING

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ABSTRACT

It is well known that an ν–th order Volterra filter of one dimensional signal can be evaluated by an appropriate ν–D linear convolution. This work describes new superfast algorithm for Volterra filtering. New approach is based on the superfast discrete Radon and Nussbaumer Polynomial Transforms.

1. INTRODUCTION

The study of nonlinear operators $y(t) = H\{x(t)\}$ was started by Volterra [1] who investigated analytic operators and introduced the representation

$$y(t) = \int_{-\infty}^{\infty} h_1(t_1-\tau_1)x(\tau_1))d\tau_1 +$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(t_1-\tau_1,t_2-\tau_2)x(\tau_1)x(\tau_2)d\tau_1 d\tau_2 + \quad (1)$$

$$+....+$$

$$+\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} h_\nu(t_1-\tau_1,\ldots,t_\nu-\tau_\nu)x(\tau_1)\cdots x(\tau_\nu)d\tau_1\cdots d\tau_\nu,$$

where $\nu = 1,2,\ldots,$ $x(t)$ and $y(t)$ are the input and output respectively of the system at time t and $h_1(t_1-\tau_1,\ldots,t_\nu-\tau_\nu)$ is the ν–th order Volterra kernel. Equation (1) is also known as a Volterra series. Such a functional representation characterises a system as a mapping between its input and output spaces. Another way of expressing it is

$$y(t) = H_1\big[u(t)\big] + H_2\big[u(t)\big] + \ldots + H_\nu\big[u(t)\big] + \ldots = \quad (2)$$

$$= y^{(1)}(t) + y^{(2)}(t) + \ldots + y^{(\nu)}(t) + \ldots,$$

in which $y^\nu := H_\nu\big[x(t)\big] =$

$$= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} h_\nu(t_1-\tau_1,\ldots,t_\nu-\tau_\nu)x(\tau_1)\cdots x(\tau_\nu)d\tau_1\cdots d\tau_\nu. \quad (3)$$

The Volterra series has been successfully applied to a wide variety of engineering problems such as modeling nonlinear communication channels and biological systems, linearizing audio speakers, and acoustic noise cancellation.

If the product of expression (3) is interpreted as a ν–D signal $x(\tau_1,\tau_2,\ldots,\tau_\nu) := x(\tau_1)x(\tau_2)\cdots x(\tau_\nu)$, expression (3) can be seen as a multidimensional convolution evaluated on the main diagonal $t = t_1 = \ldots = t_\nu$:

$$y^{(\nu)}(t_1,t_2,\ldots,t_\nu)\Big|_{t=t_1=t_2=\ldots=t_\nu} =$$

$$= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} Y(f_1,\ldots,f_\nu)e^{2\pi j(t_1f_1+\ldots+t_\nu f_\nu)}df_1\cdots df_\nu = \quad (4)$$

$$= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} Y(f_1,\ldots,f_\nu)e^{2\pi j(f_1+f_2+\ldots+f_\nu)t}df_1\cdots df_\nu,$$

where $Y(f_1,\ldots,f_\nu) := H_\nu(f_1,\ldots,f_\nu)X(f_1)\cdots X(f_\nu)$, $H_\nu(f_1,\ldots,f_\nu)$ and $X(f)$ are the Fourier transforms of $y(t)$, $h_\nu(t_1,\ldots,t_\nu)$ and $x(t)$, respectively.

This interpretation suggest the use of FFT based schemes, also called fast convolution schemes, for the computation of (1). These methods are the most efficient techniques for evaluating linear convolutions. Their application to Volterra filtering in principle is rather attractive because the direct computation of (3) requires a number of operations per output point of order of $N^{\nu+1}$, while the computation of (2) via fast convolution requires a $\nu N^{\nu-1}$ 1–D Fast Fourier Transforms (FFT's) or $\nu N^\nu \log N$ arithmetic operations, where N is the number of samples along one dimension. This paper describes a superfast new ν–D Fast Fourier Transform. It requires fewer 1–D FFT's than the classical separable radix–2 FFT–type approach. The method utilizes a decomposition of the ν–D Fourier transform into a product of (original) ν–D Discrete Radon Transform and a minimal family parallel/independ 1–D Fourier Transforms. In this case our approach leads to decrease of multiplicative complexity by factor of ν compared to the classical row/column separable approach.

Note that none of the multidimensional FFT algorithms for this application reported in literature can at once calculate the signal from its Fourier spectrum on the main diagonal. Only composition Radon transform and a collection parallel/independed 1–D FFT can calculate signal on this diagonal.

In order to develop a superfast nonlinear convolution we need the Radon transform (RT). This transform and its ill–conditioned inverse were first formulated by J. Radon in 1917. Currently, the RT is used in a wide variety of applications including tomography, ultrasound, optics, and geophysics, to name a few. In this paper, we introduce new direct and inverse DRT and show that they admit fast computation by the fast Nussbaumer Polynomial Transform (NPT) [6].

2. MULTIDIMENSIONAL RADON AND FOURIER TRANSFORMS

Let R^ν be a ν–D space consisting of column vectors $\mathbf{x} := (x_1, x_2, \ldots, x_\nu)^* = |\mathbf{x}\rangle$ with components $x_1, x_2, \ldots, x_\nu$ over the field of real numbers in orthonormal basis $\{e_i\}_{i=1}^\nu$, where "*" denotes transpose. Let $R^{*\nu}$ be the dual space consisting of row vectors $\omega := (\omega_1, \omega_2, \ldots \ldots, \omega_\nu) = \langle\omega|$ in the dual basis $\tilde{\mathbf{e}}_{i=1}^\nu$.

Definition 1 The unitary operators $\mathcal{F}_\nu$ and $\mathcal{F}_\nu^{-1}$ acting by rules

$$\mathcal{F}_\nu\{f(\mathbf{x})\} := \frac{1}{\sqrt{(2\pi)^\nu}} \int\limits_{R^\nu} f(\mathbf{x}) e^{j2\pi\langle\omega|\mathbf{x}\rangle} d\mathbf{x} = F(\omega), \quad (5)$$

$$\mathcal{F}_\nu^{-1}\{F(\omega)\} := \frac{1}{\sqrt{(2\pi)^\nu}} \int\limits_{R^{*\nu}} F(\omega) e^{-j2\pi\langle\omega|\mathbf{x}\rangle} d\omega = f(\mathbf{x}) \quad (6)$$

are called *direct and inverse ν–D Fourier transforms* (FT), where

$$\langle\mathbf{x}|\omega\rangle := \sum_{i=1}^{\nu} x_i\omega_i,$$

$$d\mathbf{x} := dx_1 \ldots dx_\nu, \quad d\omega := d\omega_1 \ldots d\omega_\nu.$$

Denote by $\Sigma_{\nu-1} \times \mathbf{R}^+$ the space of all hyperplanes $\pi_\nu(p)$: $\langle\xi, \mathbf{x}\rangle = p$, where $\xi \in \Sigma_{\nu-1}$, $p \in R^+$ and $\Sigma_{\nu-1}$ is $(\nu-1)$–D unit sphere in $R^{*\nu}$.

Definition 2 *The Radon transforms* of the functions $f(\mathbf{x})$ and $F(\omega)$ are the functions $\widehat{f}(\xi, p)$ and $\widehat{F}(\xi, p)$, respectively, on $\Sigma_{\nu-1} \times \mathbf{R}^+$ given by formulas

$$\widehat{\Re}_\nu\{f(\mathbf{x})\} := \widehat{f}(\xi, p) = \int\limits_{R^\nu} f(\mathbf{x})\delta(p - \langle\xi|\mathbf{x}\rangle)\, d\mathbf{x}, \quad (7)$$

$$\widehat{\Re}_\nu\{F(\omega)\} := \widehat{F}(\xi, p) = \int\limits_{R^\nu} F(\omega)\delta(p - \langle\xi|\omega\rangle)\, d\omega, \quad (8)$$

i.e. $\widehat{f}(\xi, p)$and $\widehat{F}(\xi, p)$ are equal to integrals of the function $f(\mathbf{x})$ and $F(\omega)$ along hyperplanes $\pi_\nu(p)$.

Radon transform is closely related with ν–D Fourier Transform

$$f(\mathbf{x}) = \int\limits_{R^{*\nu}} F(\omega) e^{j2\pi\langle\omega|\mathbf{x}\rangle} d\omega.$$

Indeed, if $\mathbf{x} = |\mathbf{x}|\xi^o = t\xi^o$, where $\xi^o \in \Sigma_{\nu-1} \subset R^\nu$, $t = |\mathbf{x}|$, then to calculate $f(\mathbf{x}) = f(t\xi^o)$, one can first make integration by hyperplane $\langle\omega|\xi^o\rangle = p$, and then the integrate 1–D Fourier transforms by p (for every fixed ξ^o) : $f(\mathbf{x}) = f(t\xi^o) =$

$$= \mathcal{F}_\nu^{-1}\{F(\omega)\} = \int\limits_{R^\nu} F(\omega) e^{j2\pi t\langle\omega|\xi^o\rangle} d\omega =$$

$$= \int\limits_{-\infty}^{+\infty} \left(\int_{\langle\omega|\xi^o\rangle = p} F(\omega) d\omega \right) e^{j2\pi tp} dp =$$

$$= \int\limits_{-\infty}^{+\infty} \widehat{F}(\xi^o, p) e^{j2\pi tp} dp := \mathcal{F}_1^{-1}\left\{\widehat{F}(\xi^o, p)\right\}, \quad (9)$$

where $\widehat{F}(\xi^o, p) =$

$$= \int\limits_{\langle\omega|\xi^o\rangle = p} F(\omega) d\omega = \int\limits_{\mathbf{R}^\nu} F(\omega)\delta(\langle\omega|\xi^o\rangle - p) d\omega.$$

It means that the ν–D FT $\mathcal{F}_\nu^{-1}$ is a composition of the Radon transform $\Re_\nu$ and of a family 1–D FT

$$\mathcal{F}_\nu^{-1} = \left\{ {}^{\xi^0}\mathcal{F}_1^{-1} \mid \xi^o \in \Sigma_{\nu-1} \right\} \odot \Re_\nu.$$

The cardinality of 1–D Fourier transforms from this the family $\left\{ {}^{\xi^0}\mathcal{F}_1^{-1} \mid \xi \in \Sigma_0 \right\}$ equal to the cardinality of the sphere (set) $\Sigma_{\nu-1}$. This cardinality is finite in discrete case and generate minimal complexity ν–D FFT.

Consider $\Sigma_{\nu-1} \times R^+$ as a subset of $\mathbf{R}^{\nu+1}$. It is clear that it forms a cylinder like surface with p as the axis of the cylinder. We can consider $\widehat{F}(\xi^0, p)$ as defined on this surface. Consider the $f(t\xi^0)$ as the inverse Fourier transform of the spectrum $F(\omega)$. Now $t\xi^0$ is a point on the ray generated by the unit vector ξ^0 and is obtained from $\widehat{F}(\xi^0, p)$ by 1–D Fourier transform of function $\widehat{F}(\xi^0, p)$ by variable p. Since rays $t\xi^0$ generated by $\xi^0 \in \Sigma_{\nu-1}$ cover signal domain R^ν then all values ν–D signal can be obtained. This connection is called *Central Slice Theorem.*

Theorem 1 1–D Fourier transform ${}^{\xi^0}\mathcal{F}_1^{-1}$ of projection $\widehat{F}(\xi^0, p)$ along the ray $t\xi^o$ is the central slice of ν–dimensional signal $f(t\xi^0)$.

3. RADON TRANSFORM AND VOLTERRA SERIES

Return to the expression (4) for $\nu = 2$

$$y^{(2)}(t_1, t_2) = y(\mathbf{t}) = \int\limits_{\infty}^{\infty}\int\limits_{\infty}^{\infty} Y^{(2)}(f_1, f_2) e^{2\pi j(t_1 f_1 + t_2 f_2)} df_1 df_2 \quad (10)$$

Introduce 2–D "time" (t_1, t_2)–space $\mathbf{R}^2$ spanned by two unit "time" vectors $\mathbf{e}_1, \mathbf{e}_2$. Arbitrary 2–D unit vector $\mathbf{t}^0 = t_1^0\mathbf{e}_1 + t_2^0\mathbf{e}_2 \in \Sigma_2$ will be called *arrow of the time* (in terms of the physic & phylosophy). Let $\mathbf{t} = (t_1, t_2)$ be an arbitrary 2–D time vector. Then $\mathbf{t} = |\mathbf{t}|\left(\frac{t_1}{|\mathbf{t}|}, \frac{t_2}{|\mathbf{t}|}\right) = t_s\mathbf{t}^0$, where $\mathbf{t}^0 := \left(\frac{t_1}{|\mathbf{t}|}, \frac{t_2}{|\mathbf{t}|}\right)$, $t_s := |\mathbf{t}|$ is the *eigentime* lying on $\mathbf{t}^0$–th arrow of time. Classical physical (calendar) time t_{ph} lie on the main diagonal $t_1 = t_2$, i.e.on the ray $\mathbf{t} = \frac{t_s}{\sqrt{2}}(1, 1) = t_{ph}(1, 1)$, where $t_{ph} := t_s/\sqrt{2}$.

Definition 3 The arrow of time $\mathbf{t}_{ph}$ is called *arrow of physical (calendar) time* and all the other $\mathbf{t}^0 \neq \mathbf{t}_{ph}^0$ are called *arrows ghost time.*

Represent expression (10) as

$$y^{(2)}(\mathbf{t}) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Y^{(2)}(f_1,f_2)e^{2\pi j\langle\mathbf{t}|\mathbf{f}\rangle}df_1df_2 =$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Y^{(2)}(f_1,f_2)e^{2\pi j(t_1f_1+t_2f_2)}df_1df_2.$$

Let $\mathbf{t} = |\mathbf{t}|\mathbf{t}_0 = t_s\mathbf{t}^0$, where $\mathbf{t}^0 \in \Sigma_2$, then to calculate $y^{(2)}(t_s\mathbf{t}^0)$, one can first make integration by line $\langle\mathbf{t}^0|\mathbf{f}\rangle = f = \text{const}$, and then integrate 1–D Fourier transform by f (for every fixed $\mathbf{t}^0$–th arrow of time):

$$y^{(2)}(t_s\mathbf{t}^0) = \int\limits_{\infty}^{\infty}\int\limits_{\infty}^{\infty} Y^{(2)}(f_1,f_2)e^{2\pi jt_s\langle\mathbf{t}^0|\mathbf{f}\rangle}df_1df_2 =$$

$$= \int\limits_{\infty}^{\infty}\left(\int\limits_{\langle\mathbf{t}^0|\mathbf{f}\rangle} Y^{(2)}(f_1,f_2)df_1df_2\right)e^{2\pi jt_sf}df =$$

$$= \int\limits_{-\infty}^{\infty}\widehat{Y}^{(2)}(\mathbf{t}^0,f)e^{2\pi jt_sf}df,$$

where $\widehat{Y}^{(2)}(\mathbf{t}^o,f) := \int\limits_{\langle\mathbf{t}^o|\mathbf{f}\rangle=f} Y^{(2)}(f_1,f_2)df_1df_2 =$

$$= \int\limits_{\mathbf{R}^2} Y^{(2)}(f_1,f_2)\delta(f-t_1^of_1+t_2^of_2)df_1df_2.$$

Obviously the physical output of quadratic filter lies on the main diagonal $t_1 = t_2$, i.e. on the arrow of the physical time $\mathbf{t}^o_{ph}$: $y^{(2)}(t) = y^{(2)}(t_{ph}\mathbf{t}^o_{ph}) =$

$$= y^{(2)}(\sqrt{2}t_s\mathbf{t}^0_{ph}) = \int\limits_{-\infty}^{\infty}\widehat{Y}^{(2)}(\mathbf{t}^0_{ph},f)e^{2\pi jt_sf}df,$$

where $\widehat{Y}^{(2)}(\mathbf{t}^o,f) :=$

$$= \int\limits_{\mathbf{R}^2} Y^{(2)}(f_1,f_2)\delta\left(f-\frac{1}{\sqrt{2}}f_1+\frac{1}{\sqrt{2}}f_2\right)df_1df_2.$$

Definition 4 The output of quadratic filter $y^{(2)}(t) = y^{(2)}(\sqrt{2}t_s\mathbf{t}^0_{ph})$ lying on the arrow of the physical time $\mathbf{t}^o_{ph}$ is called *physical output* and all the other outputs $y^{(2)}(\sqrt{2}t_s\mathbf{t}^0)$, $\mathbf{t}^o \neq \mathbf{t}^o_{ph}$ are called *ghost outputs* or $\mathbf{t}^o$th–outputs.

Definition 5 A some k–collection of the $\mathbf{t}^o$th–outputs of quadratic filter $\mathbf{VG}^{(2)}(t) :=$

$$= \begin{bmatrix} y^{(2)}(t_s\mathbf{t}^0_1) \\ y^{(2)}(t_s\mathbf{t}^0_2) \\ \dots \\ y^{(2)}(t_s\mathbf{t}^0_k) \end{bmatrix}$$

is called *Volterragram of the 2–th order.*

The idea using of 2–D Radon Transform for calculation of output quadratic filter can be generalized further. Return to the expression (4). Introduce ν–D "time" $\mathbf{t} := (t_1,\dots,t_\nu)$–space $\mathbf{R}^\nu$. Denote by $\Sigma_{\nu-1} \in \mathbf{R}^\nu$ unit sphere. Arbitrary unit vector $(t_1^o,\dots,t_\nu^o) \in \Sigma_{\nu-1}$ is called an arrow of the time. For an arbitrary ν–D vector $\mathbf{t} := (t_1,\dots,t_\nu)$ we have $\mathbf{t} := |\mathbf{t}|\mathbf{t}^o = t_s\mathbf{t}^o$, where $t_s =: |\mathbf{t}|$ is eigentime lying on the arrow of the time $\mathbf{t}^o := \left(\frac{t_1}{\sqrt{\nu}},\dots,\frac{t_\nu}{\sqrt{\nu}}\right) = (t_1^o,\dots,t_\nu^o)$. Physical time t_{ph} lies on the main diagonal $t_1 = t_2 = \dots = t_\nu$, i.e. on the ray $\mathbf{t} = \frac{t_s}{\sqrt{\nu}}(1,\dots,1) = t_{ph}(1,\dots,1)$, where $t_{ph} := t_s/\sqrt{\nu}$.

Represent (4) as $y^{(\nu)}(\mathbf{t}) = y^{(\nu)}(t_1,\dots,t_\nu) =$

$$= \int\limits_{-\infty}^{\infty}\dots\int\limits_{-\infty}^{\infty} Y^{(\nu)}(f_1,\dots,f_\nu)e^{2\pi j\langle\mathbf{t}|\mathbf{f}\rangle}df_1\cdots df_\nu =$$

$$= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} Y^{(\nu)}(f_1,\dots,f_\nu)e^{2\pi j(t_1f_1+\dots+t_{nu}f_\nu)}df_1\cdots df_\nu.$$

Let $\mathbf{t} = |\mathbf{t}|\mathbf{t}_0 = t_s\mathbf{t}^0$, where $\mathbf{t}^0 \in \Sigma_{\nu-1}$, then calculate $y^{(\nu)}(t_1,\dots,t_\nu) = y^{(\nu)}(\mathbf{t}) = y^{(\nu)}(t_s\mathbf{t}^0)$, one can first make integration by line $\langle\mathbf{t}^0|\mathbf{f}\rangle = f = \text{const}$, and then integrate 1–D Fourier transform by f (for every fixed $\mathbf{t}^0$–th arrow of the time):

$$y^{(\nu)}(t_s\mathbf{t}^0) = \int\limits_{\infty}^{\infty}\dots\int\limits_{\infty}^{\infty} Y^{(\nu)}(f_1,\dots,f\nu)e^{2\pi jt_s\langle\mathbf{t}^0|\mathbf{f}\rangle}df_1\cdots df_\nu =$$

$$= \int\limits_{\infty}^{\infty}\left(\int\limits_{\langle\mathbf{t}^0|\mathbf{f}\rangle} Y^{(\nu)}(f_1,\dots,t_\nu)df_1\cdots df_\nu\right)e^{2\pi jt_sf}df =$$

$$= \int\limits_{-\infty}^{\infty}\widehat{Y}^{(\nu)}(\mathbf{t}^0,f)e^{2\pi jt_sf}df,$$

where $\widehat{Y}^{(\nu)}(\mathbf{t}^o,f) := \int\limits_{\langle\mathbf{t}^o|\mathbf{f}\rangle=f} Y^{(\nu)}(f_1,\dots,f_\nu)df_1\cdots df_\nu =$

$$= \int\limits_{\mathbf{R}^\nu} Y^{(\nu)}(f_1,\dots,f_\nu)\delta(f-t_1^of_1+\dots+t_\nu^of_\nu)df_1\cdots df_\nu.$$

Again the physical output of quadratic filter lies on the main diagonal $t_1 = t_2 = \dots = t_\nu$, i.e. on the arrow of the physical time $\mathbf{t}^o_{ph}$: $y^{(\nu)}(t) = y^{(\nu)}(t_{ph}) =$

$$= y^{(\nu)}(\sqrt{\nu}t_s\mathbf{t}^0_{ph}) = \int\limits_{-\infty}^{\infty}\widehat{Y}^{(\nu)}(\mathbf{t}^0_{ph},f)e^{2\pi jt_sf}df,$$

where $\widehat{Y}^{(\nu)}(\mathbf{t}^o,f) :=$

$$= \int\limits_{\mathbf{R}^\nu} Y^{(\nu)}(f_1,\dots,f_\nu)\delta\left(f-\frac{f_1+\dots+f_\nu}{\sqrt{\nu}}\right)df_1\cdots df_\nu.$$

Definition 6 The output of ν–th filter $y^{(\nu)}(\sqrt{\nu}t_s\mathbf{t}^0_{ph}) = y^{(\nu)}(t)$ lying on the arrow of the physical time $\mathbf{t}^o_{ph}$ is called *physical output* and all the other outputs $y^{(\nu)}(\sqrt{\nu}t_s\mathbf{t}^0)$, $\mathbf{t}^o \neq \mathbf{t}^o_{ph}$ are called *ghost outputs* or $\mathbf{t}^o$th–outputs.

Definition 7 A k–collection of the $\mathbf{t}^o$th–outputs ν–th order $\mathbf{VG}^{(\nu)}(t) :=$

$$= \begin{bmatrix} y^{(\nu)}(t_s\mathbf{t}^0_1) \\ y^{(\nu)}(t_s\mathbf{t}^0_2) \\ \dots \\ y^{(\nu)}(t_s\mathbf{t}^0_k) \end{bmatrix}$$

is called *Volterragram of ν–th order.*

According to (2) we have

$$y(t_s\mathbf{t}_0) = H_1\big[u(t)\big] + H_2\big[u(t)\big] + \dots + H_\nu\big[u(t)\big] + \dots =$$

$$= \int_{-\infty}^{\infty} Y^{(1)}(f)e^{2\pi j t_s f}df + \int_{-\infty}^{\infty} \widehat{Y}^{(2)}(\mathbf{t}^0, f)e^{2\pi j t_s f}df +$$

$$+ \dots +$$

$$+ \int_{-\infty}^{\infty} \widehat{Y}^{(\nu)}(\mathbf{t}^0, f)e^{2\pi j t_s f}df + \dots =$$

$$= \int_{-\infty}^{\infty} \widehat{Y}^{(\nu)}_{\Sigma}(\mathbf{t}^0, f)e^{2\pi j t_s f}df,$$

where

$$\widehat{Y}^{(\nu)}_{\Sigma}(\mathbf{t}^0, f) := \sum_{\nu=1}^{\infty} \widehat{Y}^{(\nu)}_{\Sigma}(\mathbf{t}^0, f)$$

is the Fourier spectrum of output signal.

Definition 8 The output of Volterra filter $y(t_{ph}\mathbf{t}^0_{ph}) = y(t)$ lying on the arrow of the physical time $\mathbf{t}^o_{ph}$ is called *physical output* and all other outputs

$$y^{\nu}(t_s\mathbf{t}^0) = \int_{-\infty}^{\infty} \widehat{Y}^{(\nu)}_{\Sigma}(\mathbf{t}^0, f)e^{2\pi j t_s f}df,$$

are called *ghost outputs* or $\mathbf{t}^o$th–outputs.

Definition 9 A k–collection of the $\mathbf{t}^o$th–outputs of Volterra filter $\mathbf{VG}_\sigma(t) :=$

$$= \begin{bmatrix} y(t_s\mathbf{t}^0_1) \\ y(t_s\mathbf{t}^0_2) \\ \dots \\ y(t_s\mathbf{t}^0_k) \end{bmatrix}$$

is called *full Volterragram* of Volterra filter.

A Volterragramm gives a full representation about all possibility output (physical and ghost) of a Volterra filter with different points of view (in different time arrow). In philosophy such approach is called *geschtalt phylosophy.*

4. DISCRETE VOLTERRA FILTERS

Discrete time version of the Volterra series is

$$y(n) = \sum_{i=0}^{N-1} h_1(n-i)x(i) +$$

$$+ \sum_{i_1=0}^{N-1}\sum_{i_2=0}^{N-1} h_2(n-i_1, n-i_2)x(i_1)x(i_2) +$$

$$+ \dots +$$

$$+ \sum_{i_1=0}^{N-1} \dots \sum_{i_\nu=0}^{N-1} h_\nu(n-i_1, \dots, n-i_\nu)x(i_1)\cdots x(i_\nu) + \dots \quad (11)$$

where $h_1(i)$, $h_2(i_1, i_2)$, $h_\nu(i_1, i_2, \dots, i_\nu)$, ... are the linear, quadratic, cubic etc. filter weights or kernels respectively and N denotes the filter length. Every term of Volterra series (11) is the output of a homogeneous and causal ν–th order Volterra filter

$$y^{(\nu)}(n) = \sum_{i_1=0}^{N-1} \dots \sum_{i_\nu=0}^{N-1} h_\nu(n-i_1, \dots, n-i_\nu)x(i_1)\cdots x(i_\nu).$$

By using discrete Fourier transform, we can write:

$$y^{(\nu)}(n) = \sum_{0}^{N-1} \dots \sum_{0}^{N-1} Y^{(\nu)}(k_1, \dots, k_\nu)e^{\frac{2\pi j}{N}(k_1+\dots+k_\nu)n},$$

where $H^{(\nu)}(k_1, \dots, k_\nu)$, $X(k)$ and $Y^{(\nu)}(k_1, \dots, k_\nu) = H^{(\nu)}(k_1, \dots, k_\nu)X(k_1)\cdots X(k_\nu)$, are the Fourier transform of $h_\nu(i_1, \dots, i_\nu)$, $x(i)$ and $y^{(\nu)}(n)$, respectively.

If the product of expression (11) is interpreted as a ν–D discrete signal $x(i_1, \dots, i_\nu) = x(i_1)\cdots x(i_\nu)$, expression (11) can be seen as a ν–D convolution evaluated on the main diagonal $n = n_1 = \dots = n_\nu$:

$$y^{(\nu)}(n_1, \dots, n_\nu)\Big|_{n=n_1=\dots=n_\nu} =$$

$$= \sum_{k_1=0}^{N-1} \dots \sum_{k_\nu=0}^{N-1} Y^{(\nu)}(k_1, \dots, k_\nu)e^{2\pi j(n_1k_1+\dots+n_\nu k_\nu)/N}, \quad (12)$$

which can be implemented efficiently by using fast Radon and Fourier transforms.

5. DISCRETE FOURIER AND RADON TRANSFORMS

Let $\mathcal{D}^\nu(N) = Z_N\mathbf{e}_1 + Z_N\mathbf{e}_2 + \dots + Z_N\mathbf{e}_\nu$ be ν–D a discrete paralellepiped. Its elements are column vectors $\mathbf{i} := (i_1, \dots, i_\nu)^* = |\mathbf{i}\rangle$, where $i_1, \dots, i_\nu \in Z_N$. Let $\mathcal{D}^{*\nu}(N)$ be a dual paralellepiped consisting of row vectors $\mathbf{k} := (k_1, k_2, \dots, k_\nu) = \langle\mathbf{k}|$, $k_1, \dots, k_\nu \in Z_N$.

Definition 10 The unitary operators $\mathcal{F}_\nu$ and $\mathcal{F}_\nu^{-1}$ acting by rules

$$\mathcal{F}_\nu\{f(\mathbf{i})\} := \frac{1}{\sqrt{N}} \sum_{\mathbf{i}\in\mathcal{D}^\nu} f(\mathbf{i})e^{\frac{-j2\pi\langle\mathbf{i}|\mathbf{k}\rangle}{N}} = F(\mathbf{k}),$$

$$\mathcal{F}_\nu^{-1}\{F(\mathbf{k})\} := \frac{1}{\sqrt{N}} \sum_{\mathbf{k}\in\mathcal{D}_N^{*\nu}} F(\mathbf{k}) e^{\frac{j2\pi\langle\mathbf{i}|\mathbf{k}\rangle}{N}} = f(\mathbf{i})$$

are called *direct and inverse discrete ν–D Fourier transforms* (DFT), where $\langle\mathbf{i}|\mathbf{k}\rangle := \sum_{s=1}^{\nu} i_s k_s$.

For convenience we omit in the following the normalization factor (constant) $1/\sqrt{N}$.

Let $\{\alpha^\circ\} \in \mathcal{D}^{*\nu}(N)$ be minimal such vector set that the rays $\{a\alpha^\circ \mid = 1,2,\dots,N-1\}$ cover the whole parallelepiped $\mathcal{D}^{*\nu}(N) : \{a\alpha^\circ \mid a = 1,2,\dots,N-1\}$. Then we can write that $F(a\alpha^0) =$

$$= \sum_{\mathbf{i}\in\mathcal{D}^\nu} f(\mathbf{i}) e^{\frac{-j2\pi a}{N}\langle\alpha^0|\mathbf{i}\rangle} = \sum_{p=0}^{q-1}\left(\sum_{<\alpha^\circ|\mathbf{i}>=p} f(\mathbf{i})\right) e^{\frac{2j\pi ap}{N}},$$

or

$$F(a\alpha^\circ) = \sum_{p=0}^{q-1} \widehat{f}(\alpha^\circ, p) e^{\frac{2j\pi ap}{N}}, \qquad (13)$$

where

$$\widehat{f}(\alpha^\circ, p) := \mathcal{R}_\nu\{f(\mathbf{i})\} = \sum_{<\alpha^\circ|\mathbf{i}>=p} f(\mathbf{i}).$$

Definition 11 The function $\widehat{f}(\alpha^\circ, p)$ which is equal to the sum of values of the signal $f(\mathbf{i})$ on the discrete hyperplane $\langle\alpha^\circ|\mathbf{i}\rangle = p$ is called *Discrete Radon Transform* (DRT) of $f(\mathbf{i})$ [7].

The expression (13) means that ν–D DFT $\mathcal{F}_\nu$ is a composition of DRT $\mathcal{R}_\nu$ and a set 1–D DFT's. The total number of 1–D DFT is equal to the power of the set $\{\alpha^\circ\}$. Every 1–D DFT acts along the ray $\{a\alpha^\circ \mid a = 1,2,\dots,L(\alpha^\circ)\}$, where $L(\alpha^\circ)$ is the length of the ray. It is necessary to find such $\{\alpha^0\}$ that would give DRT with minimum computational complexity. Note that the classical "rown/column separable " ν–D DFT is reduced to $\nu N^{\nu-1}$ 1–D DFT's of the lenght N.

Theorem 2 [9] If $N = q$ is a prime integer the total number of rays RAY(ν, q) that cover $\mathcal{D}^{*\nu}(q)$ is equal to

$$\text{RAY}(\nu, q) = (q^\nu - 1)/(q-1) \approx q^{\nu-1},$$

and each of rays has length $L(n,q) = q$. All rays spanned by the following vectors of set $\{\alpha^0\}$:

- $\{\alpha^o\}^\nu := (k_1,\dots,k_{\nu-2},k_{\nu-1},1)$,
- $\{\alpha^o\}^{\nu-1} := (k_1,\dots,k_{\nu-2},1,0)$,
- ....,
- $\{\alpha^o\}^2 := (k_1,1,0\dots,0)$,
- $\{\alpha^o\}^1 := (1,0,\dots,0)$,

where $k_i \in \mathbf{Z}_q$, $i = 1,2,\dots,n$.

First, this means that ν–D FT is realized not as $\nu q^{\nu-1}$ 1–D FT, but with help only $q^{\nu-1}$ 1–D FT, second, this mean that ν–th order discrete Volterra filter have $(q^\nu-1)/(q-1)$ arrows of the time and its Volterragram contain $(q^\nu - 1)/(q-1)$ terms.

The total complexity of the proposed algorithm [9] for computing ν- -D DRT is

$$\mathbf{Ad}(\mathcal{R}_\nu(q)) = \nu q^{\nu-2}\mathbf{Ad}(\mathcal{N}_1(q)), \mathbf{Mu}(\mathcal{R}_\nu(q)) = 0$$

and for computing ν–D DFT is

$$\mathbf{Ad}(\mathcal{F}_\nu(q)) \approx (\nu+1)q^{\nu-1}\mathbf{Ad}(\mathcal{F}_1(q)),$$

$$\mathbf{Mu}(\mathcal{F}_\nu(q)) = \frac{q^{\nu-1}-1}{q-1}\mathbf{Mu}(\mathcal{F}_1(q)),$$

where $\mathbf{Ad}(\mathcal{N}_1(q)) = q\mathbf{Ad}(\mathcal{F}_1(q))$, $\mathbf{Mu}(\mathcal{N}_1(q)) = 0$; $\mathbf{Ad}(\mathcal{N}_1(q))$, $\mathbf{Mu}(\mathcal{N}_1(q))$ and $\mathbf{Ad}(\mathcal{F}_1(q))$, $\mathbf{Mu}(\mathcal{F}_1(q))$ are additive and multiplicative complexities 1–D q–points fast NPT $\mathcal{N}_1(q)$ and DFT $\mathcal{F}_1(q)$, respectively.

6. CONCLUSIONS

Volterra non–linear convolution consists of a collection of multidimensional convolutions. Every ν–D convolution (12) can be evaluated by an classical fast convolution algorithm, which requires a $\nu N^\nu/(N+1)$ 1–D FFT's. Our method requires one ν–D Radon transform (Nussbaumer transform) and one (!) Fourier transform, for calculating physical output of Volterra filter and one ν–D Radon transform and $N^{\nu-1}$ Fourier transforms, for calculating physical and all ghost outputs of Volterra filter, that decreasing the computer complexity by the factor of ν (where ν is the order Volterra filter) compared to the classical row/column separable FFT approach.

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